Problem 1. Let (X, d) be a metric space and let $\mathcal{T}_d$ denote the metric topology induced by d and let X_d denote the topological space $(X, \mathcal{T}_d)$.

1. Prove that $d\colon X_d \times X_d \to \mathbb{R}_{std}$ is continuous.

2. Prove that if $\mathcal{T}$ is another topology on X such that $d\colon (X, \mathcal{T}) \times (X, \mathcal{T}) \to \mathbb{R}_{std}$ is continuous, then $\mathcal{T}_d$ is coarser than $\mathcal{T}$.

Note: This proves that the metric topology $\mathcal{T}_d$ is the coarsest topology on X such that the given distance function $d\colon X \times X \to \mathbb{R}$ is continuous.

Problem 2. Let (X, d) be a metric space. Define a new metric on X by

$$\overline{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for all } x, y \in X.$$

1. Verify that $\overline{d}$ is a metric.

   Hint: For the triangle inequality of $\overline{d}$, use that of d and the fact that the map $f\colon [0, \infty) \to [0, 1)$, where $f(x) = \frac{x}{1+x}$, is increasing and satisfies $f(a) + f(b) \geqslant f(a + b)$.

2. Prove that d and $\overline{d}$ induce the same topology. *Hint:* Use Problem 1, and the fact that both f and its inverse are continuous.

3. For $X = \mathbb{R}$, show that d and $\overline{d}$ are not equivalent in general.

Problem 3. Let (X, d) be a metric space and let $Y \subseteq X$ be a subset. The restriction of d to Y defines a metric on Y

$$d_Y\colon Y \times Y \to [0, \infty), \quad d_Y(x, y) = d(x, y) \text{ for all } x, y \in Y.$$

Show that the metric topology on Y associated to d_Y coincides with the induced topology of Y as a subspace of X.

Problem 4. Let (X, d_X) and (Y, d_Y) be metric spaces. Assume that $f\colon X \to Y$ is a function such that

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \qquad \text{for all } x_1, x_2 \in X$$

1. Prove that f is injective.

2. Prove that $f\colon X \to Y$ is continuous.

3. Prove that $f\colon X \to f(X)$ is open, where $f(X) \subseteq Y$ has the induced topology.

Note: This proves that the corestriction $f\colon X \to f(X)$ is a homeomorphism. In this case, f is said to be an imbedding of X into Y.

Problem 5. Let X be a topological space and consider $X \times X$ with the product topology. Prove that X is Hausdorff if and only if the *diagonal* $\Delta = \{(x, y) \in X \times X \mid x = y\}$ is closed in $X \times X$.