8. Friday, July 5th. Homeomorphisms (§18) Metric Spaces (§20, 21)

Last Time: Continuous maps. Restriction and corestriction.

Homeomorphism: continuous bijective function with continuous inverse.
Not automatic!

Definition 1: Two Top spaces X and Y are homeomorphic if there is a homeomorphism $f: X \to Y$.

Proposition 2: Let $f: X \to Y$ be a bijective map between Top spaces. Then $f^{-1}: Y \to X$ is continuous $\Longleftrightarrow$ $f(U) \subseteq Y$ is open for every open $U \subseteq X$.

Proof: HW!

Definition 3: $f: X \to Y$ is open if $f(U) \subseteq Y$ is open for every open $U \subseteq X$.

Examples:

1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax+b$ with $a \neq 0$ is continuous (analysis) and bijective with inverse $f^{-1}: \mathbb{R} \to \mathbb{R}$, $f^{-1}(y) = \frac{y-b}{a}$ also continuous.

2) Consider $(-1,1) \subseteq \mathbb{R}$ with the induced topology

The map $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = \dfrac{x}{1+|x|}$ is continuous (analysis), injective and $g(\mathbb{R}) = (-1,1)$. Hence $g|^{(-1,1)}_{\mathbb{R}} \to (-1,1)$ is continuous and bijective.

The inverse map $h: (-1,1) \to \mathbb{R}$ given by $h(y) = \dfrac{y}{1-|y|}$ is also continuous (analysis)

Hence $\mathbb{R}$ and $(-1,1)$ are homeomorphic.

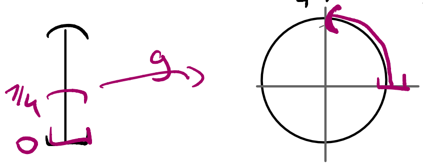
3) Let $f: \mathbb{R} \to \mathbb{R}^2$ where $f(t) = (\cos(2\pi t), \sin(2\pi t))$, continuous by analysis

Note that $f|_{[0,1)} : [0,1) \to \mathbb{R}^2$ is injective with image $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

Hence f gives a bijective continuos map $g: [0,1) \to S^1$, $g(t) = (\cos(2\pi t), \sin(2\pi t))$

However we will see that g is <u>not</u> a homeomorphism, because g is <u>not</u> open

In fact let $U = [0, \frac{1}{4})$, open in $[0,1)$



$g(U) = \{(x,y) \in S^1 \mid x > 0, y \geq 0\}$ is not open (To be proved)

Metric Spaces

Crucial for $\mathbb{R}_{STD}$: $B(a,\varepsilon) = \{ x \in \mathbb{R} \mid \underbrace{|x-a|}_{\text{distance from } x \text{ to } a} < \varepsilon \}$

We have a distance function $d(x,y) = |x-y|$

Definition 4: Let X be a set. A metric (or distance) on X is a function

$d: X \times X \longrightarrow [0,\infty)$ such that, $\forall x,y,z \in X$,

1) $d(x,y) = 0 \iff x = y$ (non degeneracy)
2) $d(x,y) = d(y,x)$ (symmetry)
3) $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle inequality)

A pair (X,d) where d is a metric on X is called a metric space

If (X,d) is a metric space, for each $a \in X$ and $\varepsilon > 0$ we denote

$B_d(a,\varepsilon) = \{ x \in X \mid d(a,x) < \varepsilon \}$. The open ball of radius ε centered at a.

Also denote $B(a,\varepsilon)$ if d is clear.

Examples

1) $X = \mathbb{R}$ with $d(x,y) = |x-y|$.

2) $X = \mathbb{R}^n$ with Euclidean metric $d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$

More generally, for each $p \in [1, \infty)$ we have a metric

$$d_p(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^p\right)^{1/p}$$

The Triangle inequality for d_p follows from Minkowski's inequality (analysis)

3) For $p = \infty$ we have

$$d_\infty(x,y) = \max\{ |x_i - y_i| \mid 1 \leq i \leq n \}$$

4) Any (non empty) set can be endowed with the discrete Metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Proposition 5: Let (X,d) be a metric space. Then the collection of open balls

$$\mathcal{B} = \{ B(x,\varepsilon) \mid x \in X,\ \varepsilon > 0 \}$$

form a basis for a topology on X.

Definition: The topology $\mathcal{T}_d = \mathcal{T}(\mathcal{B})$ is called the metric topology on X.

Proof: We use Proposition 3.4. The first condition $X = \bigcup_{B \in \mathcal{B}} B$ is trivial because for any $x \in X$ we have $x \in B(x,1)$.

To prove the second condition, we first prove:

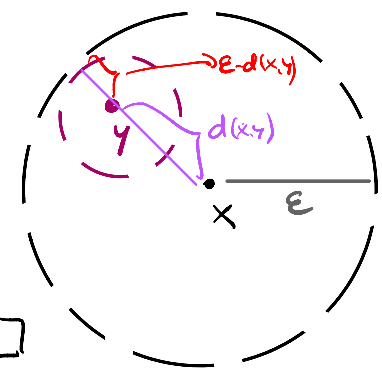
Lemma: Let $B(x,\varepsilon) \in \mathcal{B}$. Then, $\forall y \in B(x,\varepsilon)$, $\exists \delta > 0$ such that $B(y,\delta) \subseteq B(x,\varepsilon)$

Proof: Since $y \in B(x,\varepsilon)$, we know that $\varepsilon - d(x,y) > 0$

Take any δ with $0 < \delta < \varepsilon - d(x,y)$.

We prove $B(y,\delta) \subseteq B(x,\varepsilon)$. Let $z \in B(y,\delta)$. Then

$$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \delta < d(x,y) + \varepsilon - d(x,y) = \varepsilon \quad \square$$



Back to the proposition, let $B(x_1, \varepsilon_1)$ and $B(x_2, \varepsilon_2) \in \mathbb{B}$.

Let $y \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$. By the lemma above,

$\exists\, \delta_1 > 0$ such that $B(y, \delta_1) \subseteq B(x_1, \varepsilon_1)$ and $\exists\, \delta_2 > 0$ such that $B(y, \delta_2) \subseteq B(x_2, \varepsilon_2)$

Hence, for $\delta = \min(\delta_1, \delta_2)$, we have

$$B(y, \delta) \subseteq B(y, \delta_1) \cap B(y, \delta_2) \subseteq B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2).$$

Thus, by Prop. 3.4, $\mathbb{B}$ is a basis for some topology. $\square$

<u>Note</u>: Using the local description of $\mathcal{T}(\mathbb{B}) = \mathcal{T}_d$, we know that

$$\mathcal{T}_d = \{ U \subseteq X \mid \forall x \in U \; \exists\, \varepsilon > 0 \text{ such that } B(x, \varepsilon) \subseteq U \}.$$