Last time: For a a graph $G = (V, E)$, we defined:

(1) The **independence number** $\alpha(G)$ is the size of a largest independent set in G.

(2) The **edge-independence number** $\alpha'(G)$ is the size of a largest matching in G.

(3) A **vertex-cover** of G is a subset $U \subset V$ such that every edge of G has at least one end-point in U. The **vertex-covering number** $\beta(G)$ is the size of a smallest vertex-cover of G.

(4) An **edge-cover** of G is a subset $S \subset E$ so that every vertex of G is incident to at least one edge in S. The **edge-covering number** $\beta'(G)$ is the size of a smallest edge-cover of G.

Example.

Today:

- $\alpha'(G) \leqslant \beta(G) \leqslant 2\alpha'(G)$ for any graph.
- $\alpha'(G) = \beta(G)$ if G is bipartite.
- $\alpha'(G) + \beta'(G) = |V(G)|$ if G has no isolated vertices.
- $\alpha(G) + \beta(G) = |V(G)|$ for any graph (homework).

Theorem. *For any graph* G *we have* $\alpha'(G) \leqslant \beta(G) \leqslant 2\alpha'(G)$.

Note: It is possible that $\beta(G) = 2\alpha'(G)$, just take $G = K_n$ with n odd.

Next: $\alpha'(G) = \beta(G)$ is also possible.

Theorem (König's Theorem)**.** *If* G *is a bipartite graph, then* $\alpha'(G) = \beta(G)$*. In other words, for bipartite graphs, the size of a maximum matching is equal to the size of a smallest vertex-cover.*

Lemma. *Let* $G = (V, E)$ *be a graph with no isolated vertices. If* $S \subset E$ *is a minimum edge-cover of* G*, then* (V, S) *is a forest with no isolated vertices.*

Theorem (8.7)**.** *If* G *is a graph on* n *vertices none of which is isolated, then* $\alpha'(G) + \beta'(G) = n$*.*

Theorem (8.8)**.** *If* G *is a graph on* n *vertices, then* $\alpha(G) + \beta(G) = n$*.*

Example. *Compute* $\alpha(P)$ *and* $\beta(P)$ *where* P *is the Petersen graph.*

Next: a nice consequence of Hall's theorem.

Theorem. *Let* G *be a bipartite graph with parts* U, W *such that* $|U| \leqslant |W|$*. Assume that* G *has no isolated vertices and that* $\deg u \geqslant \deg w$ *whenever* $u \in U$ *and* $w \in W$ *are such that* $uw \in E(G)$*. Then* G *has a matching which saturates* U*.*