Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts)**.** Let G and H be graphs. A *graph homomorphism* from G to H is a function $f\colon V(G) \to V(H)$ such that $\{f(u), f(v)\} \in E(H)$ whenever $\{u, v\} \in E(G)$. Prove that a graph G is bipartite if and only if there is a graph homomorphism from G to K_2.

(Note that a graph homomorphism does *not* need to be a bijection and that it could be the case that $\{f(u), f(v)\} \in E(H)$ even though $\{u, v\} \notin E(G)$)

Problem 2 (2 pts)**.** A graph G is called *self-complementary* if $G \cong \overline{G}$. For example, P_4 and C_5 are self-complementary.

Prove that if G is a self-complementary graph on n vertices, then n is congruent to either 0 or 1 modulo 4.

Problem 3 (2 pts)**.** Let T be a tree on n vertices with $\Delta(T) \leqslant 2$. Prove that $T \cong P_n$.

Problem 4 (2 pts)**.** Determine (with proof) all trees T (up to isomorphism) on $n \geqslant 2$ vertices such that $\overline{T}$ is also a tree. (Note: we do not require that $T \cong \overline{T}$.)

Problem 5 (2 pts)**.** Let G and H be two self-complementary graphs on disjoint vertex sets, where H has even order n. Let F be the graph obtained from $G \cup H$ by joining each vertex of G to every vertex of degree less than $n/2$ in H. Show that F is self-complementary.

Hint: A graph F is self-complementary if there exists a $\varphi : V(F) \to V(F)$ such that $vw \in E(F)$ if and only if $\varphi(v)\varphi(w) \notin E(F)$.