Last week:

- Matchings
- Edge- and vertex-covers
- Edge- and vertex- independence and covering numbers $\alpha, \alpha', \beta, \beta'$.

This week:

- Vertex coloring
- Chromatic numbers

Definition. *Let* G *be a graph.*

- *A* ***coloring*** *of* G *is a function* $f\colon V(G) \to C$ *where* C *is a non-empty set.*
- *A* ***proper coloring*** *of* G *is a coloring* $f\colon V(G) \to C$ *such that if* $uv \in E(G)$*, then* $f(u) \neq f(v)$*.*
  In this case C *is called the* ***palette*** *and the elements in* C *are* ***colors****.*
- *For an integer* $t \geqslant 1$*, a* t***-coloring*** *of* G *is a proper coloring* $f\colon V(G) \to C$ *such that* $|C| \leqslant t$*.*
- *The* ***chromatic number*** $\chi(G)$ *is the smallest integer* t *such that* G *has a* t*-coloring.*

Example.

Quick observations:

(a) If G is an n-vertex graph, then $\chi(G) \leqslant n$. Also, $\chi(G) = n$ if and only if $G \cong K_n$.

(b) $\chi(G) = 1$ if and only if G has no edges.

(c) If H is a subgraph of G, then $\chi(H) \leqslant \chi(G)$.

Definition. *The* ***clique-number*** $\omega(G)$ *of a graph* G *is the largest size of a clique contained in* G.

(d) $\chi(G) \geqslant \omega(G)$

Definition. *A **color class** of a proper coloring* $f\colon V(G) \to C$ *is the set* $f^{-1}(c)$ *for any* $c \in C$.

Proposition. *Proper colorings of* G *using* t*-colors correspond to partitions of* $V(G)$ *into* t *independent sets.*

In particular, $\chi(G)$ *is the smallest integer* t *for which* G *is* t*-partite.*

Theorem (10.5)**.** *If* G *is an* n*-vertex graph, then*

$$\chi(G) \geqslant \frac{n}{\alpha(G)}$$

Example ($\chi(G) = \omega(G)$ and $\chi(G) = \frac{n}{\alpha(G)}$)**.**

Definition. *For a graph* G*, the* ***degeneracy*** *of* G *is*

$$d(G) = \max\{\delta(H) \colon H \textit{ is a subgraph of } G\}$$

Theorem (10.7, 10.9)**.** *For any graph* G*, we have* $\chi(G) \leqslant 1 + d(G)$*. In particular,* $\chi(G) \leqslant 1 + \Delta(G)$*.*

Theorem (Brook's, 10.8). *For a connected graph* G *that is not an odd cycle or a clique, we have* $\chi(G) \leqslant \Delta(G)$.

Example.