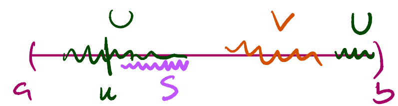
19. Wednesday, July 19. Connectedness in $\mathbb{R}$

Theorem 1: Every open interval $(a,b) \subseteq \mathbb{R}$ is connected.

Proof: Assume for a contradiction $(a,b) = U \cup V$ is a separation of (a,b) where $U, V \subseteq (a,b)$ are open. Since $U \neq \emptyset$ and $V \neq \emptyset$, there are points $u \in U$, $v \in V$ which are $u \neq v$ because $U \cap V = \emptyset$. Assume wlog $u < v$.

Consider the subset $S \subseteq (a,b)$ given by

$$S = \{ s \in (a,b) \mid [u, s] \subseteq U \}$$



Note: $S \neq \emptyset$ (because $u \in S$) and S admits an upper bound, because $S \subseteq (a,b)$.
Therefore, S admits a supremum $s_0 = \sup S$.

Claim 1. Every $x \in V$ with $u < x$ is an upper bound for S.

Proof: Assume for a contradiction this is not the case. i.e, there is some $x \in V$ and some $s \in S$ with $x < s$.

Then $x \in (u, s) \subseteq [u, s] \subseteq U$, so $x \in U \cap V$, contradiction

Claim 2: $s_0 = \sup S \in (a,b)$

Proof: $[a<s_0]$ Since $u \le s_0$ and $u \in U \subseteq (a,b)$, we have $a < u \le s_0$, hence $a < s_0$.

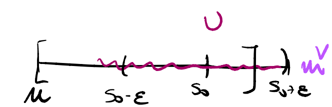
$[s_0 < b]$. Since $v \in V$ and $u < v$, we know that v is an upper bound for S (by Claim 1). Therefore $s_0 \le v$. But $v \in V \subseteq (a,b)$, hence $s_0 \le v < b$.

Claim 3-ε: $[u, s_0) \subseteq U$.

In fact, otherwise, there is some $x \in V$ with $u < x < s_0$, hence x is an upper bound for S (by Claim 1), but then $x < s_0 = \sup S$, contradiction.

Claim 3: $s_0 \notin U$.

Proof: For a contradiction, if $s_0 \in U$, since U is open, we have $(s_0 - \varepsilon, s_0 + \varepsilon) \subseteq U$ for some $\varepsilon > 0$. Then, by claim 3-ε, we have



$$\left[u, s_0 + \frac{\varepsilon}{2}\right] \subseteq U,$$ Therefore $s_0 + \frac{\varepsilon}{2} \in S$, which contradicts that s_0 is an upper bound.

Claim 4: $s_0 \notin V$.

Proof: If $s_0 \in V$, then $(s_0 - \varepsilon_1, s_0 + \varepsilon_1) \subseteq V$ for some $\varepsilon_1 > 0$ because V is open

Also, $u < s_0$, so $u < s_0 - \varepsilon_2$ for some ε_2. For $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, we get $(s_0 - \varepsilon, s_0 + \varepsilon) \subseteq V$ and $u < s_0 - \varepsilon$.

Hence $s_0 - \frac{\varepsilon}{2} \in V$ and $u < s_0 - \frac{\varepsilon}{2}$, hence $s_0 - \frac{\varepsilon}{2}$ is an upper bound for S by Claim 1. This contradicts s_0 is the smallest one. $\square$

Corollary 2: $\mathbb{R}$ is connected, and so are the intervals (a,b), $[a,b)$, $(a,b]$, $(-\infty, a)$, $(-\infty, a]$ (a, ∞), $[a, \infty)$.

Proof: We know that $(-\infty, a)$, (a, ∞) and $\mathbb{R}$ are all homeomorphic to (a,b).
For the others, use that if A is connected then any B with $A \subseteq B \subseteq \bar{A}$ is also connected. $\square$

Intermediate Value Theorem: Let $f: X \to \mathbb{R}$ be a continuous map with X connected, and let $a, b \in X$ with $f(a) < f(b)$.
Then, for every r with $f(a) < r < f(b)$, there is some $c \in X$ with $f(c) = r$.

Proof: Assume there is no $c \in X$ with $f(c) = r$, so $r \notin f(X)$. Then

$$f(X) = \left(f(X) \cap (-\infty, r)\right) \cup \left(f(X) \cap (r, \infty)\right)$$

gives a separation of $f(X)$, which contradicts the fact that X is connected and f continuous $\square$

Path-connectedness

Definition: Let X be a top space and let $x, y \in X$. A path in X from x to y is a continuous map $\gamma : [0,1] \longrightarrow X$ with $\gamma(0) = x$, $\gamma(1) = y$.

We say that X is path connected if for any $x, y \in X$ there is a path from x to y.

Theorem 3: If a space X is path-connected, then it is connected.

Proof: Assume for a contradiction X is not connected and take a separation $X = U \cup V$. Fix $x \in U$, $y \in V$. Then there is a path $\gamma : [0,1] \longrightarrow X$ from x to y. Then $\gamma^{-1}(U), \gamma^{-1}(V)$ separate X:

In fact, these are open in $[0,1]$ because γ is continuous, and

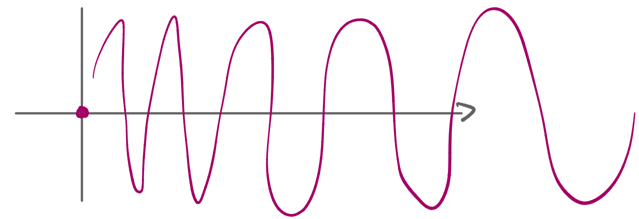
$$[0,1] = \gamma^{-1}(X) = \gamma^{-1}(U \cup V) = \gamma^{-1}(U) \cup \gamma^{-1}(V)$$

$$\gamma^{-1}(U) \cap \gamma^{-1}(V) = \gamma^{-1}(U \cap V) = \gamma^{-1}(\emptyset) = \emptyset$$

with $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$. This contradicts $[0,1]$ connected $\square$

<u>Example</u> (The converse is not true) A space can be connected but <u>not</u> path-connected.

Let $X = \{(0,0)\} \cup \underbrace{\{(t, \sin(\frac{1}{t})) \mid 0 < t \leq 1\}}_{A}$



With the induced topology of $\mathbb{R}^2$

- A is connected, because it is the image of the continuous map

$$F : (0,1] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto (t, \sin(\tfrac{1}{t}))$$

- $\{(0,0)\}$ is a cluster point of A, so $A \subseteq X \subseteq \bar{A} \Rightarrow X$ connected.

However, X is not connected.