Recall: Let G be a graph

- A **vertex-cut** is a subset $S \subset V(G)$ such that $G - S$ is disconnected.
- The **vertex-connectivity** $\kappa(G)$ is the minimum cardinality of a vertex-cut.
- For $G = K_n$, we set $\kappa(G) = n - 1$ (because it contains no vertex-cuts).
- If k is a non-negative integer, we say that G is k-connected if $\kappa(G) \geqslant k$.

Today: computing $\kappa(G)$ using paths.

Definition. *Fix* $u, v \in V(G)$.

- *A* u–v ***separating set*** *is a vertex cut* $S \subset V(G)$ *such that* u *and* v *belong to different components of* $G - S$.
- *A* ***minimum*** u–v ***separating set*** *is a* u–v *separating set with minimum cardinality.*
- *A collection* $\{P_1, \ldots, P_k\}$ *of* u–v *paths is* ***internally disjoint*** *if every two of these paths have no vertices in common other than* u *and* v.

Note: If S is a minimum u–v separating set, there are **at most** $|S|$ internally disjoint u–v paths.
Next: actually there are exactly $|S|$.

Theorem (Menger's theorem 5.16). *Let u and v nonadjacent vertices in a graph G. The minimum number of vertices in a u–v separating set equals the maximum number of internally disjoint u–v paths in G.*

Case 1: Some minimum u–v separating set S contains a vertex x adjacent to both u and v.

Case 2: Some minimum u–v separating set S contains a vertex non-adjacent to u and a vertex non-adjacent to v.

Case 3: For every minimum u–v separating set S, either every vertex in S is adjacent to u and not v, or every vertex in S is adjacent to v and not u.

Theorem (Withney's theorem 5.17)**.** *Let* G *be a non-trivial graph and* $k \geqslant 2$ *an integer.*
G *is* k*-connected iff there are at least* k *internally disjoint* u–v *paths in* G *for each pair* u, v *of distinct vertices.*

Exercise. *Verify that* $\kappa(H_{r,n}) = r$ *by showing that every two vertices* u, v *in* $H_{r,n}$ *are connected by* r *internally disjoint* u–v *paths for the following values of* r *and* n: *(a)* $r = 3$ *and* $n = 8$. *(b)* $r = 4$ *and* $n = 8$.

Theorem (5.18)**.** *Let* G *be a* k*-connected graph and let* R *be any set of* k *vertices. If a graph* H *is obtained from* G *by adding a new vertex* w *and joining* w *to the vertices on* R, *then* H *is also* k*-connected.*

Corollary (5.19)**.** *If* G *is a* k*-connected graph and* $u, v_1, v_2, \ldots, v_k$ *are distinct vertices of* G, *then there exist a family* $\{P_1, \ldots, P_k\}$ *where* P_i *is a* u-v_i *path and for* $i \neq j$, *the paths* P_i *and* P_j *have only* u *in common.*

Recall. Theorem 5.7: A graph is 2-connected iff every two vertices lie on a common cycle.

Theorem (5.20)**.** *If* G *is a* k*-connected graph,* $k \geqslant 2$*, then every* k*-vertices of* G *lie on a common cycle of* G*.*

Edge-connectivity analogues:

Theorem (Menger's theorem for edge-connectivity 5.21). *For distinct vertices* u, v *in a graph* G*, then*

$$\textit{minimum number of } \textbf{\textit{edges}} \textit{ that separate } u \textit{ and } v = \textit{maximum number of pairwise } \textbf{\textit{edge}}\textit{-disjoint } u\text{–}v \textit{ paths}$$

Theorem (Whitney's theorem for edge-connectivity 5.22). *Let* G *be a a graph and* $k \geqslant 2$ *an integer.* G *is* k-***edge****-connected iff* G *contains* k *pairwise* ***edge-****disjoint* u–v *paths for any* $u \neq v \in V(G)$.