Today: Fix a nonempty set V. Let $\mathcal{T}_V$ denote the set of all trees on vertex-set V.

Theorem (Cayley's formula). *If* $|V| = n \geqslant 2$, *then* $|\mathcal{T}_V| = n^{n-2}$.

Note: We are counting all trees on V, not trees up to isomorphism.

The strategy is the following:
Step 0: consider $V^{n-2} = \{(x_1, x_2, \ldots, x_{n-2}) : x_i \in V\}$, all $(n-2)$-tuples in V, which has $|V^{n-2}| = |V|^{n-2} = n^{n-2}$.

Step 1: construct a function $\text{Prüfer}_V : \mathcal{T}_V \longrightarrow V^{n-2}$.

Step 2: show that Prüfer_V is a bijection from $\mathcal{T}_V$ to V^{n-2}. This will imply Cayley's formula.

To define Prüfer_V, fix an ordering on $V = \{v_1, v_2, \ldots, v_n\}$ so that $v_1 < v_2 < \cdots < v_n$.

Definition (Prüfer codes). *For an ordered set* V *with* $|V| = n \geqslant 2$, *we define a function*

$$\textit{Prüfer}_V : \mathcal{T}_V \longrightarrow V^{n-2}$$

recursively on n *as follows:*

(1) *If* $n = 2$ *define* $\textit{Prüfer}_V(T) = ()$, *the empty tuple;*

(2) *If* $n \geqslant 2$, *given* $T \in \mathcal{T}_V$ *let* $\ell \in V$ *denote the smallest leaf of* T *(with respect to the ordering on* V*). Let* v *denote* ***the*** *neighbor of* ℓ *in* T *and define*

$$\textit{Prüfer}_V(T) = (v, \textit{Prüfer}_{V-\{\ell\}}(T - \ell)).$$

Informally: to build $\text{Prüfer}_V(T)$ delete the smallest leaf, write down its neighbor, and repeat until there are only two vertices left.

Example. *Compute the Prüfer codes of the following trees.*

Lemma. *Fix an ordered set* V *with* $|V| = n \geqslant 2$. *For any* $v \in V$ *and* $T \in \mathcal{T}_V$, *the entry* v *appears exactly* $\deg_T v - 1$ *times in the sequence* $\text{Prüfer}_V(T)$.

Theorem. *For any ordered set* V *with* $|V| = n \geqslant 2$*, the function* Prüfer_V *is a bijection from* $\mathcal{T}_V$ *to* V^{n-2}*.*

According to the previous theorem, to any given sequence $P \in V^{n-2}$ one can associate a unique $T \in \mathcal{T}_V$.

Algorithm: Initialize with the empty graph $F = (V, \emptyset)$ and $S = \emptyset$, and iterate the following:

(1) Let $v \in V$ the smallest element which does not appear in S nor in P.

(2) If p denotes the first entry in P, add the edge vp to F.

(3) Remove the first entry of P (so we now have a sequence of length one fewer) and add v to S.

After $n-2$ iterations, P is the empty sequence and $|S| = n-2$. Thus $S = V - \{u, v\}$. Adding the edge uv to F yields the desired sequence.

Example. *Find the tree associated to sequence* $P = (1, 1, 1, 1, 3) \in \{1, 2, 3, 4, 5, 6, 7\}^5$.

Example. *Find the tree associated to sequence* $P = (2, 2, 3, 8, 4, 8) \in \{1, 2, 3, 4, 5, 6, 7, 8\}^6$.