Problem 1. Prove that composition of continuous functions is continuous.

Problem 2. Let $f\colon X \to Y$ be a function between topological spaces. Prove that the following are equivalent

1. f is continuous
2. $f^{-1}(C) \subseteq X$ is closed for every closed set $C \subseteq Y$.
3. If $\mathcal{S}$ is a subbasis for the topology of Y, then $f^{-1}(S) \subseteq X$ is open for every $S \in \mathcal{S}$.

Problem 3. Let $\{(X_i, \mathcal{T}_i) \mid i \in \Lambda\}$ be a collection of topological spaces and let $(X, \mathcal{T})$ be a topological space. Prove that a function $f\colon (X, \mathcal{T}) \to \left(\prod_{i\in\Lambda} X_i, \mathcal{T}_{\text{prod}}\right)$ is continuous if and only if, for all $j \in \Lambda$, the coordinate function $f_j\colon (X, \mathcal{T}) \to (X_j, \mathcal{T}_j)$ is continuous.[1]

Problem 4. We say that a function $f\colon X \to Y$ between topological spaces is *open* if $f(U) \subseteq Y$ is open for every open $U \subseteq X$.

1. Assume that $\mathcal{B}$ is a basis for the topology of X. Prove that f is open if and only if $f(B) \subseteq Y$ is open for every $B \in \mathcal{B}$.
2. Assume that f is bijective. Prove that f is open if and only if $f^{-1}\colon Y \to X$ is continuous.

Problem 5. Let $f\colon X \to Y$ be a continuous function between topological spaces. Let $A \subseteq X$ and $B \subseteq Y$ be subspaces. Prove **two** of the following.

1. The induced topology on A is the coarsest topology on A such that the inclusion $\iota_A\colon A \to X$ is continuous.
2. The restriction $f|_A\colon A \to Y$ is continuous.
3. If $f(X) \subseteq B$, then corestriction $f|^B\colon X \to B$ is continuous.

Problem 6. Let $f\colon X \to Y$ be a function between topological spaces. Assume that $\{U_i \mid i \in \Lambda\}$ is a collection of open sets in X such that $X = \bigcup_{i\in\Lambda} U_i$. Prove that f is continuous if and only if, for all $i \in \Lambda$, the restriction $f|_{U_i}\colon U_i \to Y$ is continuous.

[1]This means that the product topology is better than the box topology from a categorical perspective.