7. Wednesday, July 3rd. **Continuous Maps, Homeomorphisms** (§18)

Recall: Cartesian product of sets $X_1 \times X_2 \times \dots X_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in X_i \ \forall i \in \{1, \dots, n\}\}$

For each $j \in \{1, \dots, n\}$ have a projection $X_1 \times \dots \times X_n \xrightarrow{P_j} X_j$ where $P_j(a_1, \dots, a_n) = a_j$

More general: Product of a family $\{X_i \mid i \in \Lambda\}$ is

$$\prod_{i \in \Lambda} X_i = \{ a : \Lambda \to \bigcup_{i \in \Lambda} X_i \mid a(i) \in X_i \ \forall i \in \Lambda\}$$

For each $j \in \Lambda$ have projection $P_j : \prod_{i \in \Lambda} X_i \longrightarrow X_j$ where $P_j(a) = a(j)$

Q: How do we describe a function $f : X \longrightarrow \prod_{i \in \Lambda} X_i$ (X arbitrary set)

Definition 1: Let $f : X \longrightarrow \prod_{i \in \Lambda} X_i$ be a function. For each $j \in \Lambda$ we define the **j-th coordinate** of f as the function $f_j : X \xrightarrow{f} \prod_{i \in \Lambda} X_i \xrightarrow{P_j} X_j$ ie: $f_j = P_j \circ f$.

Recall: Product Topology on $\prod_{i\in\Lambda} X_i$ has subbasis

$$\mathcal{S} = \{ p_j^{-1}(U_j) \mid j \in \Lambda,\ U_j \text{ open in } X_j \}$$

The Product Topology Theorem: Assume $\{(X_i, \mathcal{T}_i) \mid i \in \Lambda\}$ is a collection of Top. spaces

1) The product Topology is The coarsest Topology on $\prod_{i\in\Lambda} X_i$ such That all projections $p_j : \prod_i X_i \to X_j$ are continuous.

2) Let $(X, \mathcal{T})$ be a Top. space. Then a function $f : (X, \mathcal{T}) \to \left(\prod_{i\in\Lambda} X_i, \mathcal{T}_{Prod}\right)$ is continuous if and only if The coordinate $f_j : X \to X_j$ is continuous $\forall j \in \Lambda$.

Proof: 1) Each $p_j : \left(\prod_i X_i, \mathcal{T}_{Prod}\right) \to (X_j, \mathcal{T}_j)$ is continuous because $\forall\, U_j \in \mathcal{T}_j$, we have $p_j^{-1}(U_j) \in \mathcal{S} \subseteq \mathcal{T}_{Prod}$.

Now assume $\mathcal{T}$ is a Topology on $\prod_i X_i$ such That $p_j : \left(\prod_{i\in\Lambda} X_i, \mathcal{T}\right) \to (X_j, \mathcal{T}_j)$ is cont. for all $j \in \Lambda$.

WTS: $\mathcal{T}_{Prod} \subseteq \mathcal{T}$. Since $\mathcal{T}_{Prod}$ is the coarsest topology containing $\mathcal{S}$, it's enough to show $\mathcal{S} \subseteq \mathcal{T}$.

Fix $j \in \Lambda$ and $U_j \in \mathcal{T}_j$. Since $p_j : \left(\prod_{i \in \Lambda} X_i, \mathcal{T}\right) \longrightarrow (X_j, \mathcal{T}_j)$ is continuous, we have $p_j^{-1}(U_j) \in \mathcal{T}$. Hence $\mathcal{S} \subseteq \mathcal{T}$, as desired

2) HW! Use the cont. map Thm. $\square$

<u>Example</u>: The map $f : \mathbb{R} \longrightarrow \mathbb{R}^{\mathbb{N}} = \prod_{i \in \mathbb{N}} \mathbb{R}$ where $f(t) = (t, t, \ldots)$ is continuous with the product topology. Enough to show each $f_j : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, but $f_j : \mathbb{R}_{std} \to \mathbb{R}_{std}$ is just $f_j(t) = t$, the identity! which is continuous.

Homeomorphisms

Definition 2: Let $f: X \to Y$ be a map between topological spaces. We say that f is a homeomorphism if f is continuous, bijective, and the inverse map $f^{-1}: Y \to X$ is continuous.

Note: in linear algebra if a linear map is bijective then the inverse is automatically linear no longer the case here.

Example: For any set, the identity map $Id_X: (X, T_{disc}) \to (X, T_{Triv})$ is continuous. However, the inverse $Id_X: (X, T_{Triv}) \to (X, T_{disc})$ is not continuous if X has more than one element.

In fact, if $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $\{x_1\} \in T_{disc}$ but $id_X^{-1}(\{x_1\}) = \{x_1\} \notin T_{Triv} = \{\emptyset, X\}$.

Induced Topology

Definition 3: Let (X, T_X) be a top. space. The induced topology on a set $A \subseteq X$ is

$$T_A = \{U \subseteq A \mid \exists V \in T_X \text{ with } U = A \cap V\}.$$

In this case we say that A is a subspace of X.

Example: Let $X = \mathbb{R}_{std}$. Let $A = [0, \infty)$ and $U = [0,1)$ with the induced topology. So $U \subseteq A \subseteq X$.
Note that U is open in A, because $[0,1) = [0,\infty) \cap (-1,1)$ with $(-1,1)$ open in $\mathbb{R}$.
However U is not open in $\mathbb{R}$, since it is not a neighborhood of $0 \in U$.

Definition: Let $A \subseteq X$ be a subset.

The inclusion map $\iota : A \longrightarrow X$ is defined by $\iota(a) = a \quad \forall a \in A$.

Proposition 4: Assume X is a topological space and $A \subseteq X$ is a subset.
Then the induced topology on A is the coarsest topology such that the inclusion $\iota : A \to X$ is continuous

Definition: Let $f : X \to Y$ be a function. Let $A \subseteq X$ and $B \subseteq Y$.

1) The restriction to A is the function

$$f|_A : A \xhookrightarrow{\iota_A} X \xrightarrow{f} Y \qquad \text{i.e.: } f|_A(a) = f \circ \iota_A(a) = f(a) \quad \forall a \in A.$$

2) If $f(X) \subseteq B$, the correstriction to B is

$$f|^B : X \to B, \qquad f|^B(x) = f(x) \in B \qquad \forall x \in X.$$

Proposition 5: Let $f: X \to Y$ be a continuous map.

1) If $A \subseteq X$ is a subspace, then $f|_A : A \to Y$ is continuous.

2) If $B \subseteq Y$ is a subspace and $f(X) \subseteq B$, then $f|^B : X \to B$ is continuous.

Proof: HW!