Last time: For a graph G,

- A proper coloring is a function $f\colon V(G) \to C$ where $f(u) \neq f(v)$ if $uv \in E(G)$.
- A proper coloring using t colors corresponds to a t partition of G.
- The chromatic number $\chi(G)$ is the smallest number of colors in any proper coloring.
- Lower bounds $\chi(G) \geqslant \omega(G)$ and $\chi(G) \geqslant \frac{n}{\alpha(G)}$.

  The *clique number* $\omega(G)$ is the size of the largest clique contained in G.
- Upper bounds $\chi(G) \leqslant d(G) + 1 \leqslant \Delta(G) + 1$.

  The *degeneracy* of G is $d(G) = \max\{\delta(H)\colon H \text{ is a subgraph of } G\}$.

Next: The bound $\chi(G) \geqslant \omega(G)$ is very loose.
We say that G is **triangle-free** if it contains no copies of K_3. Equivalently, $\omega(G) \leqslant 2$.

Definition. For a graph G with $V(G) = \{v_1, \ldots, v_n\}$, the **Mycielski graph** $\mu(G)$ is obtained from G by adding:

(1) *shadow vertices* $\{u_1, \ldots, u_n\}$ and edges $u_i v_j \in E(\mu(G))$ if $v_i v_j \in E(G)$,

(2) a vertex w and edges wu_i for all shadow vertices u_i.

Example (The Grötzsch graph $\mu(C_5)$)**.**

Theorem (10.10). *For every integer $k \geqslant 3$, there exists a triangle-free graph G with $\chi(G) = k$.*

Lemma 1. *If H is triangle-free, then $\mu(H)$ is triangle free.*

Lemma 2. $\chi(\mu(H)) = \chi(H) + 1$.

Theorem. *If* G *is an* n*-vertex graph, then* $\chi(G) \cdot \chi(\overline{G}) \geqslant n$*.*

Lemma. $\omega(\overline{G}) = \alpha(G)$.

Definition. *Let t be a positive integer. A graph G is t-**critical** if:*

(1) $\chi(G) \geqslant t$, *and*

(2) $\chi(H) \leqslant t - 1$ *for any proper subgraph H of G.*

Some observations:

(1) The only 1-critical graph is K_1.

(2) The only 2-critical graph is K_2.

(3) G is 3-critical if and only if G is an odd cycle.

Proposition 3. *If a graph* G *has* $\chi(G) \geqslant t$*, then it contains a subgraph which is* t*-critical.*

Proposition 4. *If* G *is* t*-critical, then* $\chi(G) = t$ *and* $\delta(G) \geqslant t - 1$.

Recall

Theorem (4.9)**.** *Let* T *be a tree on* t*-vertices. If* G *is any graph with* $\delta(G) \geqslant t-1$*, then* G *contains a copy of* T*.*

Now we can also show:

Theorem. *Let* T *be a tree on* t*-vertices. If* G *is any graph with* $\chi(G) \geqslant t$*, then* G *contains a copy of* T*.*