Instructions for the Final Exam

- You may bring one standard 8.5x11 sheet of paper with handwritten notes. **You are not allowed to include solutions to the problems on the list below.**
- All materials covered on or after Feb 28 might be evaluated.
- The final exam will consist of **four questions selected from the list below**.
- Each question will be worth 25 pts.
- There will be a bonus question, not selected from the list below, worth 15 points and graded all-or-nothing. The maximum score overall is 100 points.

Sections 5.3, 5.4

Problem 1 (12.5 + 12.5 pts)**.** Let G be a graph and suppose that H is any spanning subgraph of G.

(a) Prove that $\lambda(G) \geqslant \lambda(H)$.

(b) Prove that $\kappa(G) \geqslant \kappa(H)$.

Problem 2 (25pts)**.** Let $G = (V, E)$ be a graph with at least one edge and fix any $e \in E$. Prove that $\lambda(G - e) \geqslant \lambda(G) - 1$.

Problem 3 (25pts)**.** Let G be a graph. The k-*th power of* G is the graph G^k which has the same vertex-set as G and $uv \in E(G^k)$ iff $d_G(u, v) \leqslant k$.

Prove that if G is a connected graph on at least $k + 1$ vertices, then G^k is k-connected.

Problem 4 (10 + 15pts)**.** Let G be a bipartite graph with parts U, W where $|U| \geqslant |W|$. Show that:

(a) $\alpha(G) \geqslant |U|$ and $\alpha'(G) \leqslant |W|$.

(b) $\alpha(G) = |U|$ if and only if $\alpha'(G) = |W|$.

Problem 5 (25pts)**.** Let G be a 5-connected graph and let u, v, and w be three distinct vertices of G. Prove that G contains two cycles C and C' that have only u and v in common but neither of which contains w.

Problem 6 (25pts)**.** Prove that if a graph G is a k-connected graph that is not complete, then G has a subdivision of $K_{2,k}$.

Sections 6.1, 6.2

Problem 7 (25pts)**.** Let G be a connected regular graph that is not Eulerian. Show that, if $\overline{G}$ is connected, then $\overline{G}$ is Eulerian. Deduce that the complement of the Petersen graph is Eulerian.

Problem 8 (25pts)**.** Use Eulerian circuits to show that if G is a connected graph wherein all vertices have even degree, then G has no bridges.

Problem 9 (25pts)**.** Let G be a 6-regular graph of order 10 and let $u, v \in V(G)$. Prove that G, $G - v$, and $G - u - v$ are all Hamiltonian.

Problem 10 (25pts)**.** Show that any connected graph contains a walk which traverses each edge exactly twice.

Hint: Eulerian circuits in multigraphs.

Problem 11 (25pts)**.** Recall that K_{n_1,n_2,n_3} is the complete tripartite graph with parts of sizes n_1, n_2, n_3. Fix any positive integer n.

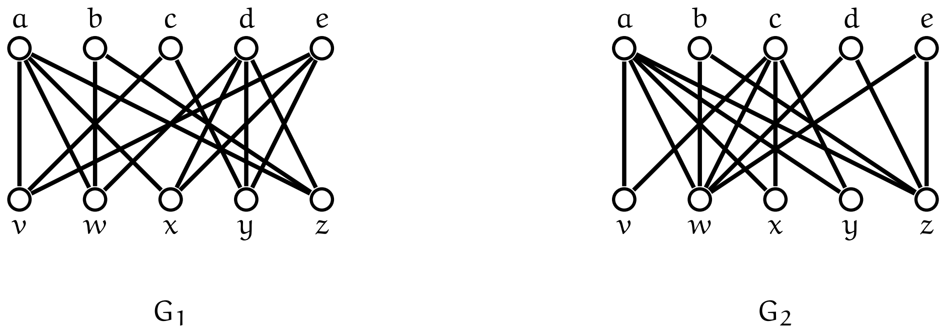
(a) Prove that $K_{n,2n,3n}$ is Hamiltonian.

(b) Prove that $K_{n,2n,3n+1}$ is not Hamiltonian.

Problem 12 (25pts)**.** Let G be a graph on $n \geqslant 4$ vertices with the property that $N(u) \cup N(v) \supseteq V(G)\backslash\{u, v\}$ for every $u \neq v \in V(G)$. Prove that G is Hamiltonian.

Sections 8.1, 8.2

Problem 13 (25pts)**.** The figure below shows two bipartite graph G_1 and G_2, each with partite sets $U = \{v, w, x, y, z\}$ and $W = \{a, b, c, d, e\}$. In each case, does the graph have a perfect matching? If so, state what the matching is; if not, explain why not.



Problem 14 (25pts)**.** Let G be a graph on $n \geqslant 2$ vertices with $\delta(G) \geqslant n/2$. Show that:

(a) If n is even, then G has a perfect matching.

(b) If n is odd, then $G - v$ has a perfect matching for every $v \in V(G)$.

Hint: Hamiltonian cycles.

Problem 15 (25pts)**.** Let G be a bipartite graph with parts U, W wherein no vertex of U is isolated. Show that if all vertices in U have distinct degrees, then G contains a matching which saturates U.

Problem 16 (25pts)**.** Let $G = (V, E)$ be a graph. Prove that if M_1 and M_2 are disjoint perfect matchings, then the graph $(V, M_1 \sqcup M_2)$ is a disjoint union of even cycles.
Hint: odd cycles are not a union of two disjoint matchings.

Problem 17 (25pts)**.** Show that the Petersen graph does not contain two disjoint perfect matchings.

Problem 18 (2pts)**.** Let G be a bipartite graph with parts U, W where $|U| = |W| = k \geqslant 1$. Prove that if $|E(G)| > k(k-1)$, then G has a perfect matching.
Hint: König's Theorem $\alpha'(G) = \beta(G)$.

Sections 10.1, 10.2

Problem 19 (25pts)**.** Let G be any graph. Prove that

$$|E(G)| \geqslant \binom{\chi(G)}{2}.$$

Problem 20 (25pts). Let $G = (V, E)$ be any graph and consider a (not necessarily proper) edge-coloring $f\colon E \to \{\text{red}, \text{blue}\})$. Let G_r and be the graph formed by the red edges and let G_b be the graph formed by the blue edges (formally, $G_r = \big(V, f^{-1}(\text{red})\big)$ and $G_b = \big(V, f^{-1}(\text{blue})\big)$). Prove that

$$\chi(G_r) \cdot \chi(G_b) \geqslant \chi(G).$$

Problem 21 (25pts). Let G be an n-vertex graph. Show that $d(\overline{G}) \leqslant n - d(G) - 1$. Here is a suggested road map:

(1) Let H be a subgraph of G with $\delta(H) = d(G)$ and let H' be a subgraph of $\overline{G}$ with $\delta(H') = d(\overline{G})$.

(2) Suppose for the sake of contradiction that $d(G) \geqslant n - d(\overline{G})$ and argue that $V(H) \cap V(H') = \varnothing$.

(3) Reach a contradiction by comparing $|V(H)|$ and $|V(H')|$.

Section 9.1

Problem 22 (25pts). Let $g \geqslant 2$ be an integer and let G be a connected plane graph on n vertices wherein every face is bounded by a cycle of G. Prove that if G has no cycles of length g or smaller, then

$$|E(G)| \leqslant \frac{g+1}{g-1}(n-2).$$

Problem 23 (25pts). Let G be a connected plane graph and suppose that every face of G has length either 5 or 6. If G is additionally 3-regular, show that G must have exactly 12 faces of length 5.

Problem 24 (25pts). Let G be a graph. Suppose G contains vertices $v_1, \ldots, v_5, \ldots$ with $\deg v_1 = 100$, $\deg v_2 = 30$, $\deg v_3 = 30$, $\deg v_4 = 4$, $\deg v_5 = 3$ and all other vertices of G have degree either 1 or 2. Knowing nothing else about G, can you determine whether or not G is planar?

Problem 25 (25pts). Is there a graph G on exactly 6 vertices which is non-planar, yet does not contain a copy of K_5 nor $K_{3,3}$?

Section 11.1

Problem 26 (25pts). Let n be any positive integer. Set $N = 2n$ if n is odd and set $N = 2n - 1$ if n is even. Show that every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of $K_{1,n}$.

Problem 27 (25pts). For an integer $N \geqslant 2$, let K_N^- denote the graph formed by deleting exactly one edge from K_N (it really doesn't matter which edge since they're all identical).

Fix any integer $n \geqslant 2$ and set $N = R(n, n)$. By definition, every red,blue-coloring of $E(K_N)$ contains a monochromatic copy of K_n. However, prove that there exists some red,blue-coloring of $E(K_N^-)$ which does not contain a monochromatic copy of K_n.

Problem 28 (25pts). Consider an even integer $k \geqslant 4$ show that

$$R_k(n_1, \ldots, n_k) \leqslant R_{k/2}(R(n_1, n_2), \ldots, R(n_{k-1}, n_k)).$$

Problem 29 (25pts). Suppose that G is a graph that has no induced $K_{1,4}$ and such that $\omega(G) = 4$. Show that $\Delta(G) \leqslant 17$.

Hint: $17 + 1 = R(4, 4)$.

Problem 30 (25pts). Show that $N = 7$ is the smallest integer N such that every red,blue-coloring of $E(K_N)$ contains either a red $K_{1,3}$ or a blue K_3.