14. Friday, July 19. Quotient spaces: Examples (§22)

Recall: Two topological spaces X and Y are homeomorphic if there is a homeomorphism $X \to Y$. We denote $X \simeq Y$

Exercise: This defines an equivalence relation!

Recall also product topology on $X \times Y$ has basis.

$$\mathcal{B}_{prod} = \mathcal{B}_{box} = \{ U \times V \subseteq X \times Y \mid U \text{ open in } X, \ V \text{ open in } Y \}$$

Exercise: Let $f : X \to X'$ and $g : Y \to Y'$ continuous maps

Then the product map

$$h : X \times Y \to X' \times Y' \quad \text{is continuous.}$$
$$(x,y) \mapsto (f(x), g(y))$$

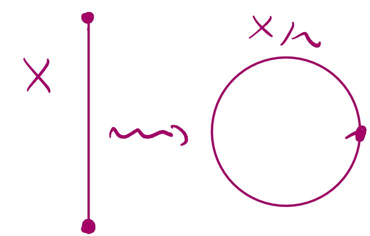
Recall also that we know how to get maps $X/\sim \xrightarrow{\bar{f}} Y$ using maps $X \xrightarrow{f} Y$. Can even decide when is $\bar{f}$ injective, surjective, continuous, open.

Example 1: (lecture 12) Let $X = [0,1]$. Define an equivalence relation by

$x \sim y \iff x = y$ or $x, y \in \{0,1\}$ The classes for this relation are

$[0,1]/_{\sim} = \{ \{x\} \mid x \in (0,1) \} \cup \{ \{0,1\} \}$

Also denoted $[0,1]/_{\sim} = [0,1]/_{\{0,1\}}$



We showed: The map $f : [0,1] \longrightarrow S^1$, $f(t) = (\cos(2\pi t), \sin(2\pi t))$

induces a continuous bijection

$\bar{f} : [0,1]/_{\sim} \longrightarrow S^1$.

Fact: $\bar{f}$ is open, so it is a homeomorphism!

We could prove this with some work, but instead we will prove this with little effort when we study compact spaces.

Recall also (from lecture 8) that the same rule

$g : [0,1) \longrightarrow S^1$, $g(t) = (\cos(2\pi t), \sin(2\pi t))$

gives a bijective continuos map

Fact: There is <u>no</u> homeomorphism $[0,1) \longrightarrow S^1$

(we sketched an argument for g and will give a rigurous map using compact spaces)

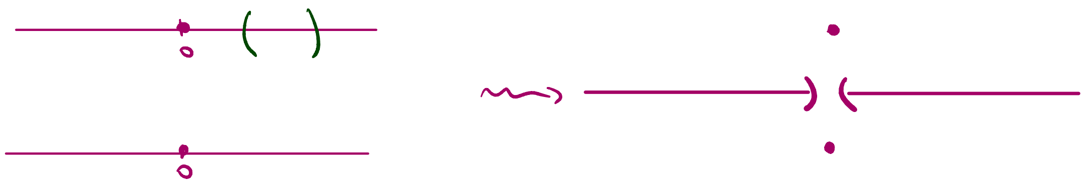
In other words, $[0,1]/\sim$ is (topologically) different from $[0,1)$

So quotient spaces do <u>not</u> delete points, they really identify points.

<u>Example 2</u>: The real line with two origins.

Let $X = \mathbb{R} \times \{-1, 1\}$ with the induced topology from $(\mathbb{R}^2, \mathcal{T}_{std})$

Define an equiv. relation on X by $(x,-1) \sim (x,1)$ for all $x \neq 0$



"The line with two origins" is $Y = X/\sim$, where the "two origins" are the classes $[(0,-1)]$ and $[(0,1)]$.

Claim 1: Y is first countable. Use $\pi: X \longrightarrow Y$

Hint: Show that $\pi((a,b) \times \{1\}) \subseteq Y$ is open for any interval (a,b)

- If $0 \notin (a,b)$, then

$$\pi^{-1}(\pi((a,b) \times \{1\})) = (a,b) \times \{-1, 1\}$$

- On the other hand, if $0 \in (a,b)$, then

$$\pi^{-1}(\pi((a,b) \times \{1\})) = (a,b) \times \{1\} \cup (a,b) \setminus \{0\} \times \{-1\}$$

Claim 2: Y is not Hausdorff.

We prove that a sequence converges to two points

Since $x_n = (\frac{1}{n}, 1) \longrightarrow (0,1)$ in X and π is continuous, we know

that $\pi(x_n) \longrightarrow \pi(0,1)$ in Y.

Similarly, $y_n = (\frac{1}{n}, -1) \longrightarrow (0,-1)$ in X, so $\pi(y_n) \longrightarrow \pi(0,-1)$ in Y

However, $x_n \sim y_n$ $\forall n$, so $\pi(x_n) = \pi(y_n)$ $\forall n$. So this sequence converges to two different points

Example 3: The Torus $S^1 \times S^1$

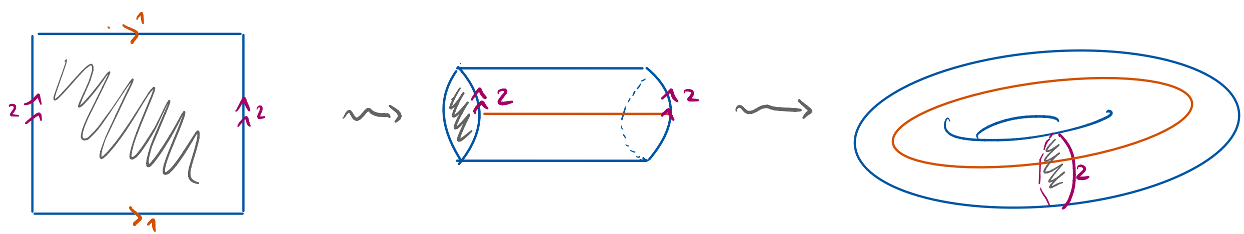
Let $X = [0,1] \times [0,1] \subseteq \mathbb{R}^2$ with the standard topology.

Define an equivalence relation on X by

$$(x,y) \sim (x',y') \text{ iff } \left(x = x' \text{ and } \{y,y'\} = \{0,1\}\right) \text{ or } \left(y = y' \text{ and } \{x,x'\} = \{0,1\}\right)$$

Alternatively, $(x,0) \sim (x,1) \ \forall x \in [0,1]$ and $(0,y) \sim (1,y) \ \forall y \in [0,1]$

Here is where pictures become veeery useful

$X =$ 

Claim: $X/\sim$ is homeomorphic to $S^1 \times S^1$

Recall from Example 1 we have

- A quotient map $q : [0,1] \longrightarrow [0,1]/\{0,1\}$

- A homeomorphism $[0,1]/\{0,1\} \xrightarrow{\bar{f}} S^1$

We take two copies of q and get a map

$$h : [0,1] \times [0,1] \longrightarrow [0,1]/_{\{0,1\}} \times [0,1]/_{\{0,1\}}$$

$$(x, y) \overset{h}{\longmapsto} (q(x), q(y))$$

Which is continuous because it is the product of two continuous maps!

Further, h descends to the quotient $[0,1] \times [0,1]/\sim$

In fact, since $q(0) = q(1)$, we have

$$h(x, 0) = h(x, 1) \quad \text{and} \quad h(0, y) = h(1, y)$$

Thus we get a continuous map

$$\bar{h} : [0,1]\times[0,1]/\sim \longrightarrow [0,1]/\sim \times [0,1]/\sim$$

- $\bar{h}$ is injective, because $h(x,y) = h(x',y') \iff (x,y)\sim(x',y')$
- $\bar{h}$ is surjective, because so is h.

Again, using compact spaces, this will imply that

$\bar{h}$ is a homeomorphism

So $[0,1]\times[0,1]/\sim \;\approx\; [0,1]/_{\{0,1\}} \times [0,1]/_{\{0,1\}} \approx S^1 \times S^1$