Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** Let G be a regular graph. Show that $\chi'(G) = \Delta(G)$ if and only if G is 1-factorable.

Problem 2 (2pts)**.** Let G be a bipartite graph with parts U, W where $|U| = |W| = k \geqslant 1$. Prove that if $|E(G)| > k(k-1)$, then G has a perfect matching.

Hint: König's Theorem $\alpha'(G) = \beta(G)$.

Problem 3 (2pts)**.** Let G be a bipartite graph with parts U, W and fix an integer $k \geqslant 1$. Let $U = U_1 \sqcup \cdots \sqcup U_k$ and $W = W_1 \sqcup \cdots \sqcup W_k$ be any partitions (some of the U_i's or W_j's may be empty). Note that $|U|$ and $|W|$ have nothing to do with k; k is just the number of pieces in each partition.

Prove that if $|E(G)| < k$, then there is a bijection $\pi\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ such that G has no edges between U_i and $W_{\pi(i)}$ for each $i \in \{1, \ldots, k\}$.

Hint: construct a bipartite graph using the U_i's and W_j's as *vertices*, and use Problem 2.

Problem 4 (2pts)**.** Let $g \geqslant 2$ be an integer and let G be a connected plane graph on n vertices wherein every face is bounded by a cycle of G. Prove that if G has no cycles of length g or smaller, then

$$|E(G)| \leqslant \frac{g+1}{g-1}(n-2).$$

Problem 5 (2pts)**.** Prove a special case of the 4-color theorem: If G is a planar, triangle-free graph, then $\chi(G) \leqslant 4$.