6. Monday, July 1st. Continuous maps (§ 18)

Last Time: $f: X \to Y$ is continuous if $f^{-1}(U) \subseteq X$ is open for all $U \subseteq Y$ open.

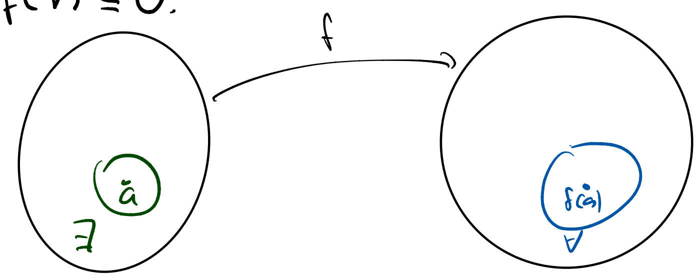
Recall from analysis: A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that if } |x-a| < \delta \text{ then } |f(x) - f(a)| < \varepsilon$$

In other words $\forall \varepsilon$, $\exists \delta$ such that $f(\underbrace{B(a,\delta)}_{\exists \text{ nbh of } a}) \subseteq \underbrace{B(f(a),\varepsilon)}_{\forall \text{ neighborhood of } f(a)}$

The Continuous Map Theorem: Let $f: X \to Y$ be a map between topological spaces.
The following are equivalent:

1) f is continuous
2) $\forall a \in X$, for every neighborhood U of $f(a) \in Y$ there exists a neighborhood V of $a \in X$ such that $f(V) \subseteq U$.



Proof: 1) ⇒ (2) Assume f is continuos. Let $a \in X$ and let $U \subseteq Y$ neighborhood of $f(a)$. Then there is some open $U' \subseteq Y$ with $f(a) \in U' \subseteq U$. We claim that $f^{-1}(U) \subseteq X$ is a neighborhood of a. In fact, $f^{-1}(U')$ is open in X (because f is continuous) and

$$a \in f^{-1}(U') \subseteq f^{-1}(U), \quad \text{as desired}$$

because $f(a) \in U'$

because $U' \subseteq U$

(2) ⇒ (1) Let $U \subseteq Y$ open. For each $a \in f^{-1}(U)$, take some open V_a of X with $a \in V_a \subseteq f^{-1}(U)$. We claim that $f^{-1}(U) = \bigcup_{a \in f^{-1}(U)} V_a$. The inclusion [⊇] follows because $V_a \subseteq f^{-1}(U)$ $\forall a$.

The inclusion [⊆] follows because, for each $a \in f^{-1}(U)$ we have $a \in V_a$, so also $a \in \bigcup_{b \in f^{-1}(U)} V_b$. $\square$

Definition 2: We say f is continuous at $a \in X$ if it satisfies Condition (2) above.

The continuous map Thm continued: The following are also equivalent:

(3) $\forall a \in X$, $\forall$ neighborhood U of $f(a) \in Y$, $f^{-1}(U)$ is a neighborhood of a in X.

(4) For every closed $C \subseteq Y$, $f^{-1}(C)$ is closed in X.

(5) For every subset $A \subseteq X$, we have $f(\bar{A}) \subseteq \overline{f(A)}$.

Some of these equivalences will be part of HW3.

Examples/Facts: Let X and Y topological spaces.

1) Any constant map $f: X \to Y$ is continuous. Indeed, if $f(x) = b \ \forall x \in X$ and $U \subseteq Y$ is open, then $f^{-1}(U) = \begin{cases} \emptyset & \text{if } b \in U \\ X & \text{if } b \notin U \end{cases}$ hence $f^{-1}(U)$ is open.

2) If X is discrete, then any map $f: X \to Y$ is continuous. In fact $f^{-1}(U) \subseteq X$ is open for any $U \subseteq Y$.

3) If Y has the trivial topology, then any map $f: X \to Y$ is continuous.
In fact opens in Y are either $\emptyset$ or Y, and $f^{-1}(\emptyset) = \emptyset$ is open, $f^{-1}(Y) = X$ is open.

4) A map $f: \mathbb{R}_{std} \to \mathbb{R}_{std}$ is continuous in our new sense iff it is continuous in the clasical sense

5) If $f: X \to Y$ and $g: Y \to Z$ are continuous then $g \circ f: X \to Z$ is continuous
In fact, if $U \subseteq Z$ is open then $(g \circ f)^{-1}(U) = \underbrace{f^{-1}(\underbrace{g^{-1}(U)}_{\text{open in } Y})}_{\text{open in } X}$

This together with the fact that $id_X : X \to X$ is continuous say that the collection of topological spaces and continuous maps form a category.

The continuous map Thm continued: The following are also equivalent:

(6) If $\mathcal{B}$ is a basis for Y, then $f^{-1}(B) \subseteq X$ is open $\forall B \in \mathcal{B}$.

(7) If $\mathcal{S}$ is a subbasis for Y, then $f^{-1}(S) \subseteq X$ is open $\forall S \in \mathcal{S}$.

Back to Cartesian products:

Example: Take $\Lambda = \mathbb{N}$ as index set. For each $i \in \mathbb{N}$, let $X_i = \mathbb{R}$ with the standard top.

We denote $\mathbb{R}^{\mathbb{N}} = \prod_{i \in \mathbb{N}} X_i = \prod_{i \in \mathbb{N}} \mathbb{R}_{std}$. We denote elements $a \in \mathbb{R}^{\mathbb{N}}$ by tuples $a = (a_1, a_2, \dots)$

Consider the map $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ where $f(t) = (t, t, \dots)$ $\forall t \in \mathbb{R}$.

On $\mathbb{R}^{\mathbb{N}}$ we have the box top. $\mathcal{T}_{box}$ and the product topology $\mathcal{T}_{prod}$.

Note that $U = \prod_{i \in \mathbb{N}} \left(-\frac{1}{i}, \frac{1}{i}\right) \subseteq \mathbb{R}^{\mathbb{N}}$ is open in the box topology (actually, basic)

However $f^{-1}(U) = \{ t \in \mathbb{R} \mid f(t) \in U \} = \{ t \in \mathbb{R} \mid t \in (-\frac{1}{i}, \frac{1}{i}) \ \forall i \in \mathbb{N} \} = \bigcap_{i \in \mathbb{N}} (-\frac{1}{i}, \frac{1}{i}) = \{0\}$

So $U \in \mathcal{T}_{box}$ but $f^{-1}(U) = \{0\}$ is not open in $\mathbb{R}$.

Hence $f : (\mathbb{R}, \mathcal{T}_{std}) \longrightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{box})$ is <u>not</u> continuous.

On the other hand $f : (\mathbb{R}, \mathcal{T}_{std}) \longrightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_{prod})$ is continuous

<u>Exercise</u>: Show that $f^{-1}(S) \subseteq \mathbb{R}$ is open $\forall S$ in the subbasis that we used to define $\mathcal{T}_{prod}$.