Recall $\deg v$ = number of edges incident with v, $\delta(G) = \min\{\deg v \colon v \in V(G)\}$, $\Delta(G) = \max\{\deg v \colon v \in V(G)\}$.
Goal: relations between vertex degrees and connectivity.

Example. *If a graph* G *in* n *vertices contains a vertex of degree* $n-1$*, then* G *is connected and* $\operatorname{diam}(G) \leqslant 2$*.*

Theorem (2.4)**.** *Let* G *be a graph in* n *vertices. If*

$$\deg u + \deg v \geqslant n - 1$$

for every two nonadjacent vertices u *and* v*, then* G *is connected and* $\operatorname{diam}(G) \leqslant 2$*.*

Note: The bound $n-1$ is sharp, i.e., there is a disconnected graph with $\deg u + \deg v \geqslant n-2$ for every two nonadjacent vertices u and v,

Corollary (2.5). *If* G *is a graph in* n *vertices and* $\delta(G) \geqslant \frac{n-1}{2}$, *then* G *is connected.*

Exercise (Exercise (2.11)). *Is the bound* $\frac{n-1}{2}$ *in the previous corollary sharp?*

Exercise (Exercise (2.9)). *Show that if* G *is a disconnected graph with exactly two odd vertices, then these odd vertices must be in the same component.*

Exercise (Exercise (2.7)). *Show that if* G *is a bipartite graph with partite sets* X *and* Y, *then* $|E(G)| = \sum_{u \in X} \deg u = \sum_{v \in Y} \deg Y$. *Can you generalize this statement for multipartite graphs?*

Definition. *A graph* G *is* ***regular*** *if all vertices have the same degree. If* $\deg v = r$ *for all* v*, we say that* G *is* ***r-regular***.

• Recall that $0 \leqslant \delta(G) \leqslant \Delta(G) \leqslant n-1$, for $n = |V(G)|$. Thus G is r-regular iff $\delta(G) = \Delta(G) = r$, where $r \in \{0, 1, \ldots, n-1\}$.

Example (All regular graphs in 4 and 5 vertices)**.**

Example (The Petersen graph)**.**

Note: If n and r are both odd, there are no r-regular graphs in n-vertices (Follows from Corollary 2.3: every graph has an even number of odd vertices).

Next: there are no further restrictions for the existence of regular graphs.

Notation: If $V(G) = \{v_1, \ldots, v_n\}$ we perform arithmetic in the subscripts modulo n.
E.g., if $n = 6$ and $i = 5$, then v_{i+2} denotes v_1.

Theorem (2.6, Harary graphs)**.** *Fix integers* r *and* n *with* $0 \leqslant r \leqslant n-1$*. There exists an* r*-regular graph in* n*-vertices if and only if at least one of* r *and* n *is even.*

Example (The Harart graphs $H_{4,10}$ and $H_{5,10}$).

Theorem (2.7)**.** *For every* G *and every integer* $r \geqslant \Delta(G)$*, there exists an* r*-regular graph* H *containing* G *as an induced subgraph.*

Example (An illustration of the proof of Theorem 2.7)**.**