10. Sequences and convergence (§17)

Definition 1: Let X be a topological space and let $a \in X$. Let $\mathcal{B}_a \subseteq \mathcal{P}(X)$ be a collection of neighborhoods of a. We say that $\mathcal{B}_a$ is a **neighborhood basis of a** if for every neighborhood U of a there exists $B \in \mathcal{B}_a$ such that $B \subseteq V$.

Examples: 1) If X is discrete and $a \in X$, then $\mathcal{B}_a = \{\{a\}\}$ is a neighborhood basis of a.

2) If X is equipped with the trivial topology, then $\mathcal{B}_a = \{X\}$ is a nbhd basis of any $a \in X$.

3) If (X, d) is a metric space and $a \in X$, then $\mathcal{B}_a = \{B(a, \varepsilon) \mid \varepsilon > 0\}$ is a nbhd basis of a. (for the metric topology $\mathcal{T}_d$).

4) If $(X, \mathcal{T})$ is a top space with a basis $\mathcal{B}$, then for each $a \in X$, the collection $\mathcal{B}_a = \{B \in \mathcal{B} \mid a \in B\}$ is a nbhd basis of a. Proof: Exercise!

Notation: A sequence in a set X is a map $\mathbb{N} \to X$ denoted by $n \mapsto x_n$.
We also use $(x_n)_{n\in\mathbb{N}}$ to denote a sequence.

Definition 2: Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a Top. space X. Let $x \in X$.

We say that (x_n) converges to x if $\forall$ open U of X with $x \in U$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $x_n \in U$.

In this case we denote $x_n \to x$ and say that x is a limit of (x_n)

Lemma 3: Let (x_n) be a sequence on a Top space X. Let $x \in X$. TFAE

1) (x_n) converges to x.

2) Assume $\mathcal{B}_x$ is a nbhd basis of x. Then $\forall\, B \in \mathcal{B}_x$ $\exists\, N \in \mathbb{N}$ such that $x_n \in B$ $\forall\, n \geq N$.

3) Assume $\mathcal{S}$ is a subbasis for X. Then for every $S \in \mathcal{S}$ with $x \in S$, $\exists\, N \in \mathbb{N}$ st $x_n \in S$ $\forall n \geq N$.

Proof: $(1) \Rightarrow (2)$ Let $B \in \mathcal{B}_x$. Since B is a nbhd of x, $\exists\, U$ open with $x \in U \subseteq B$. For this U, $\exists\, N$ such that $x_n \in U$ $\forall\, n \geq N$. Since $U \subseteq B$, we have $x_n \in B$ $\forall n \geq N$.

(2) $\Rightarrow$ (1) Let $U \subseteq X$ open with $x \in U$. Since $\mathcal{B}_x$ is nbhd basis of x, $\exists\, B \in \mathcal{B}_x$ with $x \in B \subseteq U$.

For this $B \in \mathcal{B}_x$, $\exists\, N \in \mathbb{N}$ such that $x_n \in B$ $\forall n \geq N$. Since $B \subseteq U$, we also have $x_n \in U$ whenever $n \geq N$.

(1) $\Leftarrow\Rightarrow$ (3) HW!

$\square$

Example 1: Let (X, d) metric space, (x_n) seq. in X and $x \in X$.

Since $\mathcal{B}_x = \{B(x, \varepsilon) \mid \varepsilon > 0\}$ is a nbhd basis of x, by the lemma above we know that

$$x_n \to x \iff \forall \varepsilon > 0 \ \exists\, N \in \mathbb{N} \text{ such that } x_n \in B(x, \varepsilon) \ \forall n \geq N.$$
$$\iff \forall \varepsilon > 0 \ \exists\, N \in \mathbb{N} \text{ such that } d(x, x_n) < \varepsilon \ \forall n \geq N.$$

Definition 4: A sequence (x_n) in a set X is constant if $\exists\, x \in X$ such that $x_n = x$ $\forall n \in \mathbb{N}$.

A sequence is stationary if $\exists\, x \in X$ such that $x_n = x$ $\forall n \geq N$.

Note: If X is a topological space and $(x_n) \subseteq X$ is stationary, then it converges.

Example 2: Let X be a discrete space. We claim that the only convergent sequences are the stationary ones. In fact, assume (x_n) converges to some $x \in X$. Since $\{x\}$ is open and contains x, we know $\exists\ N \in \mathbb{N}$ such that $x_n \in \{x\}\ \forall n \geq N$. Hence $x_n = x\ \forall n \geq N$.

Example 3: Endow a set X with the cocountable topology

$$T_c = \{ U \subseteq X \mid X \setminus U \text{ is countable or } U = \emptyset \}.$$

Again, the only convergent sequences are the stationary ones. In fact if (x_n) converges to x then $U = (X \setminus \{x_n \mid n \in \mathbb{N}\}) \cup \{x\}$ is open (the complement is $\subseteq \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$) and contains x. Hence $\exists\ N$ such that $x_n \in U\ \forall n \geq N$. Therefore $x_n = x\ \forall n \geq N$.

Example 4: Let X with the trivial topology. Then every sequence converges to every point. In fact, if (x_n) is a sequence and $x \in X$, the only open containing x is X, which contains $x_n\ \forall n$.

Definition 5: A topological space X is **Haussdorf** if for every $x \neq y \in X$, there are disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$

Proposition 6: Assume (x_n) is a sequence in a Haussdorf space X. If $x_n \to x$ and $x_n \to y$, then $x = y$.

Proof: Assume for a contradiction that $x \neq y$. Let U, V disjoint open sets with $x \in U$ and $y \in V$. Since $x_n \to x$, $\exists N_1$ such that $x_n \in U$ $\forall n \geq N_1$

Since $x_n \to y$, $\exists N_2$ such that $x_n \in V$ $\forall n \geq N_2$

Let $N = \max\{N_1, N_2\}$ Then $x_N \in U \cap V$, contradicts $U \cap V = \emptyset$ $\square$

Examples: 1) Any discrete space is Haussdorf.

2) Any metric space is Haussdorf.

3) If X has more than one element, the trivial topology is not Haussdorf

Note: Prop 6 says "If X is Haussdorf, then limits are unique".

Q: What about reciprocal?