Recall: For a graph $G = (V, E)$, a perfect matching is a subset $M \subseteq E(G)$ such that each vertex is incident to exactly one edge in M.

Question. If M is a perfect matching, what type of graph is (V, M)?

Example.

Definition. *A* 1*-**factor** of a graph* G *is a* 1*-regular spanning subgraph of* G.

Remark: Let $G = (V, E)$, and consider any subset $M \subseteq E$.
Then M is a perfect matching if and only if (V, M) is a 1-factor of G.

Definition. *A connected component of a graph* G *is called* ***odd*** *or* ***even*** *if it has an odd or even number of vertices, respectively. The number of odd components of* G *is denoted* $k_{odd}(G)$.

Theorem (Tutte, 8.10)**.** *A graph* G *contains a perfect matching if and only if*

$$k_{odd}(G - S) \leqslant |S| \quad \textit{for every } S \subsetneq V(G).$$

Example. *Decide if the following graph contains a perfect matching.*

Definition. *A graph G is 1-**factorable** if we can partition $E(G)$ as a disjoint union of perfect matchings.*

Theorem (8.14)**.** *The complete graph K_n for n even is 1-factorable.*

Theorem (8.15)**.** *Every regular bipartite graph is 1-factorable.*

Last time: For a graph G,

- A proper edge-coloring is a function $f\colon E(G) \to C$ where $f(e) \neq f(s)$ if e, s have a common vertex.
- A proper edge coloring using t colors corresponds to a partition of $E(G)$ into t matchings.
- The edge chromatic number $\chi'(G)$ is the smallest number of colors in any proper edge-coloring.
- Vizing's Theorem: $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.

Next: Some conditions to decide whether $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$.

Theorem (10.13)**.** *Let G be an n-vertex graph, where n is odd. If $|E(G)| > \frac{n-1}{2}\Delta(G)$, then $\chi'(G) = \Delta(G) + 1$.*

Theorem. *Let* G *be a* k*-regular graph. Then* $\chi'(G) = \Delta(G)$ *if and only if* G *is* 1*-factorable.*

Theorem (König, 10.17)**.** *If* G *is a bipartite graph, then* $\chi'(G) = \Delta(G)$.