Last week:

- Measuring connectivity:
  - Cut-vertices (delete a vertex, get more components)
  - Blocks (decompose any connected graph as a union of non-separable subgraphs which can only intersect at cut-vertices)
  - Vertex-cuts (delete a set of vertices, get more components)

This week:

- Measuring connectivity:
  - Edge-cuts
  - Vertex- and edge-connectivity
- Menger's theorem.

Recall: Let G be a graph

- A **vertex-cut** is a subset $U \subset V(G)$ such that $G - U$ is disconnected.
- The **vertex-connectivity** $\kappa(G)$ is the minimum cardinality of a vertex-cut.
- For $G = K_n$, we set $\kappa(G) = n - 1$ (because it contains no vertex-cuts).
- If k is a non-negative integer, we say that G is k-connected if $\kappa(G) \geqslant k$.

Example. *Compute* $\kappa(G)$ *for the following graphs.*

Definition. *Let* G *be a graph*

- *An **edge-cut** is a subset* $F \subset E(G)$ *such that* $G - F$ *is disconnected.*
- *The **edge-connectivity*** $\lambda(G)$ *is the minimum cardinality of an edge-cut. For* $G = K_1$*, we set* $\lambda(G) = 0$*.*
- *If* k *is a non-negative integer, we say that* G *is* k*-edge-connected if* $\lambda(G) \geqslant k$*.*

  Note: : Assume G has n vertices.

(1) $0 \leqslant \lambda(G) \leqslant n - 1$ (2) $\lambda(G) = 0$ iff G disconnected (3) $\lambda(G) = 1$ iff G has a bridge.

Example. *Compute* $\lambda(G)$ *for the following graphs.*

Proposition (5.10). *We have* $\lambda(K_n) = n - 1$ *for any positive integer* n*,*

Theorem (5.11)**.** *For every graph* G *we have* $\kappa(G) \leqslant \lambda(G) \leqslant \delta(G)$.

Example. *The bounds in Withney's theorem are sharp:*

Example. *If* G *is* r*-regular, then* $\kappa(G) \leqslant r$*.*

Corollary (5.13)**.** *If* G *has* n *vertices and* m *edges, then* $\kappa(G) \leqslant \left\lfloor \frac{2m}{n} \right\rfloor$*.*

- A **cubic** graph is a 3-regular graph

Theorem (5.12). *If* G *is a cubic graph, then* $\kappa(G) = \lambda(G)$.

Exercise. *Compute* $\kappa(G)$ *and* $\lambda(G)$ *for the Petersen graph.*

Definition. *Let* G *be a graph and* k *a positive integer. The* ***k-th power*** *of* G *is the graph* G^k *which has the same vertex-set as* G *and an edge* $uv \in E(G^k)$ *iff* $d_G(u,v) \leqslant k$.

Example.

Example (Harary graphs). *For* r, n *not both odd and* $1 \leqslant r \leqslant n-1$, *we defined* $H_{r,n}$, *an* r*-regular graph on* n*-vertices* $\{v_1, \ldots, v_n\}$.

- *If* $r = 2k$ *is even, then* $H_{r,n} = C_n^k$.
- *If* $r = 2k+1$ *is odd and* $n = 2\ell$ *is even, then* $H_{r,n} = C_n^k \cup \{v_i v_{i+\ell} \colon 1 \leqslant i \leqslant \ell\}$.

We have $\kappa(H_{r,n}) \leqslant \delta(H_{r,n}) = r$. *Can we show* $\kappa(H_{r,n}) = r$?

Exercise. G^k *is a complete graph iff diam*$(G) \leqslant k$.

Exercise. *If* G *is a connected graph on at least* $k+1$ *vertices, then* G^k *is* k*-connected.*