Definition. *A **graph** G is an ordered pair $G = (V, E)$ consisting of a set of **vertices** V and a set of **edges** E where*

$$E \subseteq \{\{u, v\} : u, v \in V, \textit{ and } u \neq v\}.$$

- Vertices of G is denoted by $V(G)$ and edges by $E(G)$.
- $|V|$ is called the **order** of G and $|E|$ is its **size**.
- If $u \neq v$ are vertices, we use uv to abbreviate $\{u, v\}$.
- If $uv \in E$, then the vertices u and v are called **adjacent** or **neighbors**.
- Two edges are **adjacent** if they share some vertex.
- A vertex v and an edge e are **incident** if $v \in e$.
- The **null graph** is the graph with $V(G) = \varnothing$.
- Two graphs G, H are **equal** if $V(G) = V(H)$ and $E(G) = E(H)$. We denote $G = H$.

Definition. *We say that a graph* H *is a* ***subgraph*** *of* G *if* $V(H) \subseteq V(G)$ *and* $E(H) \subseteq E(G)$*. We denote* $H \subseteq G$.

- H is a **proper subgraph** if $H \subseteq G$, but $H \neq \emptyset$ and $H \neq G$.
- H is a **spanning subgraph** if $H \subseteq G$ and $V(H) = V(G)$.
- H is an **induced subgraph** if $H \subseteq G$, and for all $u, v \in V(H)$, if $uv \in E(G)$ then also $uv \in E(H)$.
- If $X \subset V(G)$, then $G[X]$ denotes the induced subgraph H of G with $V(H) = X$.
- If $G_1, \ldots G_k$ are subgraphs of G such that $V(G) = V(G_1) \sqcup \cdots \sqcup V(G_k)$ and $E(G) = E(G_1) \sqcup \cdots \sqcup E(G_k)$ (disjoint unions), we say that G is the **disjoint union** of the G_i and denote $G = G_1 \sqcup \cdots \sqcup G_k$.

Definition. *A **walk** in* G *is a sequence of vertices* $W = (v_0, v_1, \dots, v_n)$ *where* $v_i v_{i+1} \in E(G)$, *for all* $0 \leqslant i < n$.

- A **trail** is a walk without repeated edges.
- A **path** is a walk without repeated vertices.
- If a {walk, trail, path} starts with u and ends with v, it is called a u–v {**walk, trail, path**}.
- The **length** of a walk is the number of edges.
- A u–v {walk, trail} is **closed** if $u = v$, and it is **open** if $u \neq v$.
- A **circuit** is a closed trail of length at least 3.
- A **cycle** is a is a circuit that repeats no vertex, except for the first and last.

Example (Identify trails, paths, circuits and cycles)**.**

Note: A **drawing** of G is a sketch assigning points to V and curves to E, where endpoints of uv are u and v. These are useful yet misleading tools, hence **your proofs shouldn't include drawings, unless explicitly requested**.

Proposition (Theorem 1.6)**.** *If a graph contains a* u–v *walk of length* ℓ*, then it contains a* u–v *path of length at most* ℓ*.*

So far, our examples share a suspicious property: we can travel from any vertex to any other vertex.

Definition. *Two vertices* u, v *in a graph* G *are said to be* ***connected*** *if* G *contains a* u–v *path.*

- For a vertex u, we allow the empty walk $W = (u)$, so that u is connected to itself.
- G itself is **connected** if every two vertices of G are connected. Otherwise, it is **disconnected**
- A **component** of G is a connected subgraph of G that is not a proper subgraph of any other connected subgraph of G.

Theorem. *Each vertex and each edge of* G *belong to exactly one component of* G*. Hence, if* $G_1, \dots G_k$ *are the components of* G*, we have* $G = G_1 \sqcup \cdots \sqcup G_k$.

Yoga: If we understand the components of a graph, we understand the graph.

Definition. *In a connected graph* G*, the* ***distance*** *between vertices* u, v *is*

$$d(u, v) := \text{ smallest length of any } u\text{–}v \text{ path in } G.$$

- A u–v path of length $d(u, v)$ is called a **geodesic**.
- The **diameter** of G, denoted $\operatorname{diam}(G)$, is the greatest distance between any two vertices.

Example (Exercise 1.12)**.** *For the depicted graph* G*, give an example of each of the following or explain why no such example exists.*

1. *An* x–y *walk of length* 6*.*
2. *A* v–w *trail that is not a* v–w *path.*
3. *An* r–z *path of length* 2*.*
4. *An* x–z *path of length* 3*.*
5. *An* x–t *path of length* $d(x, t)$*.*
6. *A geodesic whose length is* $\operatorname{diam}(G)$*.*