Last week:

- Measuring connectivity:
  - Edge-cuts
  - Vertex- and edge-connectivity
- Menger's theorem.

This week:

- Eulerian graphs
- Hamiltonian graphs

Recall: A **multigraph** might have multiple edges between any pair of vertices. Loops are not allowed.

- The degree of a vertex v in a multigraph G is the number of *edges* incident to v.
- A **walk** in a multigraph G is a collection

$$(v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k)$$

  where $v_0, v_1, \dots, v_k$ are vertices, $e_1, \dots, e_k$ are edges and the endpoints of e_i are v_{i-1} and v_i.
- The length of such walk is k, the number of edges.
- A **trail** in G is a walk that traverses each edge at most once.
- A **circuit** is a trail where the first and last vertex are equal.
- A trail with distinct endpoints is called **open**.

Example.

Definition. *Let* G *be a multigraph.*

- *An* ***Eulerian circuit*** *is a circuit in* G *that traverses each edge exactly once and uses all vertices.*
- *An* ***Eulerian trail*** *is an open trail in* G *that traverses each edge exactly once and uses all vertices.*
- *A graph is called* ***Eulerian*** *if it contains an Eulerian circuit.*

Example (Seven bridges of Königsberg)**.**

Question: Is the previous multigraph Eulerian? More generally, when is a multigraph Eulerian?
Idea: If Eulerian, every time we arrive at a vertex with an edge, we leave it with the *next* edge.
Thus every vertex should have even degree.

Lemma. *Let* G *be a multigraph, and let*

$$T = (v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k)$$

be a trail in G*. Fix* v *any vertex in* T*.*

(1) If T *is a circuit (i.e.* $v_0 = v_k$*), then* T *contains an even number of edges incident to* v*.*

(2) If T *is open and* $v \neq v_0, v_m$*, then* T *contains an even number of edges incident to* v*.*

(3) If T *is open and* $v = v_0, v_m$*, then* T *contains an odd number of edges incident to* v*.*

Lemma. *Let* G *be a multigraph where all vertices have even degree. If* v *is a non-isolated vertex, then there is a circuit in* G *using* v *and containing at least one edge.*

Theorem (6.1)**.** *Let* G *be a multigraph. Then* G *is Eulerian if and only if* G *is connected and all vertices have even degree.*

Theorem (6.2). *Let* G *be a multigraph. Then* G *has an Eulerian trail if and only if* G *is connected and has exactly two odd-degree vertices.*

Exercise. *When is* K_n *Eulerian? When is* $K_{s,t}$ *Eulerian?*

Exercise. *When does* $K_{2,t}$ *contain an Eulerian trail?*