# 4. Wednesday, June 26. Bases and subbases (SB)

Let's finish from last time

**Proposition 3.4**: Let $X$ be a set and let $\mathbb{B} \subseteq \mathcal{P}(X)$.

Then $\mathcal{T}(\mathbb{B})$ is a topology on $X$ if and only if:

1) $X = \bigcup_{B \in \mathbb{B}} B$, and

2) $\forall\, B_1, B_2 \in \mathbb{B}$ and $\forall x \in B_1 \cap B_2$, $\exists\, B_3 \in \mathbb{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$

Proof: $[\Rightarrow]$ Last time

$[\Leftarrow]$ Assume (1) and (2). WTS $\mathcal{T}(\mathbb{B})$ is a topology. We verify axioms (1), (2) and (3).

By convention, $\phi \in \mathcal{T}(\mathbb{B})$ and by (1) above $X \in \mathcal{T}(\mathbb{B})$. Also, by the first description of $\mathcal{T}(\mathbb{B})$ in Lemma 2, $\mathcal{T}(\mathbb{B})$ is automatically closed under arbitrary unions. Finally, to prove $\mathcal{T}(\mathbb{B})$ closed under finite intersections, let $U, V \in \mathcal{T}(\mathbb{B})$, say $U = \bigcup_{i \in I} B_i$ and $V = \bigcup_{j \in J} A_j$ for some $B_i, A_j \in \mathbb{B}$

WTS $U \cap V \in \mathcal{T}(\mathbb{B})$. Now $U \cap V = \bigcup_{i,j} B_i \cap A_j$. We use the second description of $\mathcal{T}(\mathbb{B})$.

Given $x \in U \cap V$, we have $x \in B_h \cap A_\ell$ for some $h, \ell$. By (2) above, $\exists B_3 \in \mathbb{B}$ with $x \in B_3 \subseteq B_h \cap A_\ell \subseteq \bigcup_{i,j} B_i \cap A_j = U \cap V$. Hence $U \cap V \in \mathcal{T}(\mathbb{B})$ by Lemma 2. $\square$

<u>Definition 1</u>: If $\mathbb{B} \subseteq \mathcal{P}(X)$ satisfies 1) and 2), we say that $\mathbb{B}$ is a basis for a topology

By proposition 3.4 it is a basis for the topology $\mathcal{T}(\mathbb{B})$.

<u>Note</u>: If $\mathbb{B} \subseteq \mathcal{P}(X)$ satisfies the condition

(2') $\forall B_1, B_2 \in \mathbb{B}$, either $B_1 \cap B_2 \in \mathbb{B}$ or $B_1 \cap B_2 = \emptyset$

Then it satisfies condition (2) in Prop 3.4 (for each $x \in B_1 \cap B_2$, take $B_3 = B_1 \cap B_2 \in \mathbb{B}$)

<u>Example</u>: In $\mathbb{R}$ we have:

1) $\mathbb{B}_{STd} = \{(a,b) \mid a < b\}$ basis for standard.

2) $\mathbb{B}_\ell = \{[a,b) \mid a < b\}$ satisfies (1) and (2), hence it is a basis for some topology, called the lower limit topology.

# Subbases

**Definition 2**: Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{S} \subseteq \mathcal{P}(X)$.

We say that $\mathcal{S}$ is a subbasis for $\mathcal{T}$ if $\mathcal{T} = \mathcal{T}(\mathbb{B}_{\mathcal{S}})$ where

$$\mathbb{B}_{\mathcal{S}} = \{ U \subseteq X \mid U = V_1 \cap V_2 \cdots \cap V_n \text{ for some } V_1, \ldots, V_n \in \mathcal{S} \}$$

i.e: open sets are arbitrary unions of finite intersections of sets in $\mathcal{S}$

**Note**: A basis for $\mathcal{T}$ is also a subbasis for $\mathcal{T}$.

Q: Given a set $X$ and $\mathcal{S} \subseteq \mathcal{P}(X)$, When is $\mathcal{S}$ a subbasis for some Topology?

**Proposition 3**: Let $X$ be a set and let $\mathcal{S} \subseteq \mathcal{P}(X)$.
Then $\mathcal{T}(\mathbb{B}_{\mathcal{S}})$ is a topology if and only if $X = \bigcup_{S \in \mathcal{S}} S$

**Proof**: Exercise, use Proposition 3.4 $\square$

**Definition 4**: Let $X$ be a set and let $\mathcal{S} \subseteq \mathcal{P}(X)$.

We say that $\mathcal{S}$ is a subbasis for a topology if $X = \bigcup_{S \in \mathcal{S}} S$.

Example: In $\mathbb{R}$, the collection $\mathcal{S} = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{(-\infty, b) \mid b \in \mathbb{R}\}$ is a subbasis for the standard topology.

## The product and the box topologies (§ 15, 19)

Recall the Cartesian product of finitely many sets: Given sets $X_1, \dots, X_n$, we define $X_1 \times X_2 \times \dots \times X_n = \{(a_1, a_2, \dots, a_n) \mid a_i \in X_i \;\; \forall i \in \{1, \dots, n\}\}$.

Here elements are tuples $a = (a_1, \dots, a_n)$, and $a_i$ is just the value of $a$ at the $i$-th entry so we can think of $a$ as a function $a : \{1, \dots, n\} \to \bigcup_i X_i$ such that $a(i) \in X_i$.

**Definition 5**: Let $\{X_i \mid i \in \Lambda\}$ be a collection of sets. We define their Cartesian product as

$$\prod_{i \in \Lambda} X_i = \left\{ a : \Lambda \to \bigcup_{i \in \Lambda} X_i \;\middle|\; a(i) \in X_i \;\; \forall i \in \Lambda \right\}$$

Note that when $\Lambda = \{1, \dots, n\}$, we recover $\prod_{i \in \Lambda} X_i = X_1 \times \dots \times X_n$

**Definition 6:** Let $\{(X_i, T_i) \mid i \in \Lambda\}$ be a collection of topological spaces.

The **box basis** is the collection

$$\mathcal{B}' = \left\{ \prod_{i \in \Lambda} U_i \subseteq \prod_{i \in \Lambda} X_i \;\middle|\; U_i \in T_i \;\; \forall i \in \Lambda \right\}$$

**Lemma/Definition 7:** $\mathcal{B}'$ is a basis for a topology on $\prod_{i \in \Lambda} X_i$, called the **box topology**

Proof: We will use Proposition 3.4. For condition (1) we need to show that the entire space $\prod_{i \in \Lambda} X_i$ is a union of sets in $\mathcal{B}'$. But since $X_i \in T_i \; \forall i \in \Lambda$, we actually have $\prod_{i \in \Lambda} X_i \in \mathcal{B}'$.
For condition (2), note that given $\prod_{i \in \Lambda} U_i \in \mathcal{B}'$ and $\prod_{i \in \Lambda} V_i \in \mathcal{B}'$, we have the set equality $\left(\prod_i U_i\right) \cap \left(\prod_i V_i\right) = \prod_{i \in \Lambda} (U_i \cap V_i)$. Since $U_i, V_i \in T_i$ and $T_i$ is a topology, we have $U_i \cap V_i \in T_i$. Hence $\left(\prod_i U_i\right) \cap \left(\prod_i V_i\right) \in \mathcal{B}'$. By Prop. 3.4, it follows that $\mathcal{B}'$ is a basis for a topology on $\prod_{i \in \Lambda} X_i$

$\square$