Last time

- A planar graph is a graph that be drawn in the plane so that edges don't cross each other.
- A plane graph is a valid drawing of a planar graph.
- Any subgraph of a planar graph is also planar.
- A plane graph G breaks the plane into a collection of connected regions, called faces of G.
- The length of a face is $\text{len}(f) = \#\{\text{edges incident to } f\} + \#\{\text{bridges incident to } f\}$.
- Euler's: If G is a connected plane graph, then $|V(G)| + |F(G)| - |E(G)| = 2$.
- Headshaking: $\sum_{f \in F(G)} \text{len}(f) = 2|E(G)|$.
- If G is planar connected on $n \geqslant 3$ vertices, then $|E(G)| \leqslant 3n - 6$. Hence K_5 is not planar.
- Moreover, if G is triangle-free then $|E(G)| \leqslant 2n - 4$.
- K_5 and $K_{3,3}$ are non-planar.

Definition. *Let* G *be a graph. A subdivision of* G *is a graph obtained by replacing some edges of* G *by paths.*

- Informally, a subdivision is obtained by inserting some new vertices into some edges.
- Each vertex introduced when creating the subdivision has degree 2.

Example.

Lemma. *Let* G *be a graph and* H *a subdivision of* H*. Then* G *is planar if and only if* H *is planar.*

Corollary. *If a graph* G *contains a subdivision of either* K_5 *or* $K_{3,3}$*, then* G *is non-planar.*

Example (Petersen)**.**

Kuratowski's Theorem, 9.7. A graph G is non-planar if and only if G contains a subdivision of either K_5 or $K_{3,3}$.

Back to coloring.

Theorem (9.3)**.** *If* G *is a planar graph, then* $\delta(G) \leqslant 5$*.*

Corollary (6-color theorem)**.** *If* G *is a planar graph, then* $\chi(G) \leqslant 6$*.*

Definition Let $f\colon V(G) \to C$ be a proper coloring of a graph G. Fix two colors $a \neq b \in C$.

- We use $G[a, b]$ to denote the induced subgraph of G containing all vertices colored by a or b.
- An $[a, b]$**-Kempe component** is a component of $G[a, b]$. We use $C[a, b]$ to denote such components.
- The **Kempe swap** associated to a Kempe component $C[a, b]$ is the coloring $f'\colon V(G) \to C$ where

$$f'(v) = \begin{cases} f(v), & \text{if } v \notin C[a, b], \\ a, & \text{if } v \in C[a, b] \text{ and } f(v) = b, \\ b, & \text{if } v \in C[a, b] \text{ and } f(v) = a. \end{cases}$$

Example.

Lemma. *The Kempe swap* f' *is a proper coloring.*

Theorem (5-color theorem)**.** *If* G *is a planar graph, then* $\chi(G) \leqslant 5$.

Step 1. Can assume wlog that G is 6-critical.

Step 2. $\delta(G) = 5$.

Step 3. Fix a vertex v with $\deg v = 5$. There is a proper coloring $f\colon V(G - v) \to \{1, \ldots, 5\}$.

Step 4. Can assume wlog that the 5 neighbors of v use distinct colors.

Step 5. Can order the neighbors of v anticlockwise: $x_1, \ldots, x_5$ and assume $f(x_i) = i$.

Step 6. If we add the 5 edges $x_i x_{i+1}$, the new graph is still planar and the coloring remains proper.

Step 7. Let $C[1,3]$ denote the $[1,3]$-Kempe component G containing x_1. Then $C[1,3]$ contains x_3.

Step 7.5 There is an x_1-x_3 path $P_{1,3}$ which uses only the colors 1 and 3 and contains no vertices from $v, x_1, \ldots, x_5$ other than x_1 and x_3.

Step 8. There is an x_2-x_4 path $P_{2,4}$ which uses only the colors 2 and 4 and contains no vertices from $v, x_1, \ldots, x_5$ other than x_2 and x_4.

Step 9. $P_{1,3}$ and $P_{2,4}$ are vertex-disjoint.

Step 10. G contains a subdivision of K_5.