**Last time:**

**Definition.** *Let* $G$ *be a multigraph.*

- *An **Eulerian circuit** is a circuit in* $G$ *that traverses each edge exactly once and uses all vertices.*
- *An **Eulerian trail** is an open trail in* $G$ *that traverses each edge exactly once and uses all vertices.*

**Lemma.** *Let* $G$ *be a multigraph, and let*

$$T = (v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k)$$

*be a trail in* $G$*. Fix* $v$ *any vertex in* $T$*.*

*(1) If* $T$ *is a circuit (i.e.* $v_0 = v_k$*), then* $T$ *contains an even number of edges incident to* $v$*.*

*(2) If* $T$ *is open and* $v \neq v_0, v_k$*, then* $T$ *contains an even number of edges incident to* $v$*.*

*(3) If* $T$ *is open and* $v = v_0, v_k$*, then* $T$ *contains an odd number of edges incident to* $v$*.*

**Theorem** (6.1)**.** *Let* $G$ *be a multigraph. Then* $G$ *is Eulerian if and only if* $G$ *is connected and all vertices have even degree.*

**Theorem** (6.2). *Let* $G$ *be a multigraph. Then* $G$ *has an Eulerian trail if and only if* $G$ *is connected and has exactly two odd-degree vertices.*

**Definition.** *Let* $G$ *be a graph.*

- *A* ***Hamiltonian cycle*** *is a cycle in* $G$ *that uses all vertices.*
- *A* ***Hamiltonian path*** *is a path in* $G$ *that uses all vertices.*
- $G$ *is a* ***Hamiltonian graph*** *if it contains a Hamiltonian cycle.*

**Example.** *Decide if the following graphs contain a Hamiltonian cycle or a Hamiltonian path.*

**Some first remarks:** Assume $G$ is a graph on $n \geqslant 3$ vertices and $C$ is a Hamiltonian cycle in $G$. Then

(a) Deleting any edge of $C$ gives a Hamiltonian path in $G$.

(b) $C \cong C_n$ is a cycle of length $n$. Thus $G$ is obtained from $C_n$ by adding some additional edges.

(c) $C \cong C_n$ contains no cycles of length less than $n$

(d) Each $v \in V(G)$ has exactly two neighbors in $C$.

(e) If $v \in V(G)$ has $\deg_G v = 2$, then both edges incident to $v$ lie on $C$.

**Example.** *The following graph is non-Hamiltonian:*

**Example.** *The Petersen graph is non-Hamiltonian:*

**Next:** Unlike Eulerianity, deciding if a graph is Hamiltonian is *hard*.
We give several sufficient conditions for Hamiltonianity.

**Theorem** (6.5). *If* $G$ *is a Hamiltonian graph, then for every nonempty proper set* $U \subsetneq V(G)$ *we have*

$$\text{Number of components of } G - U \leqslant |U|$$

**Example.** *The following graph is non-Hamiltonian*

**Corollary.** *If* $G$ *contains a cut vertex, then it is non-Hamiltonian.*

**Exercise.** *Write down the contrapositive to Theorem 6.5. Prove the Corollary*

**Theorem** (6.6, Ore's condition)**.** *Let* $G$ *be a graph on* $n \geqslant 3$ *vertices. If*

$$\deg u + \deg v \geqslant n$$

*for each pair* $u, v$ *of nonadjacent vertices, then* $G$ *is Hamiltonian.*

**Corollary** (6.7, Dirac's condition). *Let* $G$ *be a graph on* $n \geqslant 3$ *vertices. If* $\deg v \geqslant n/2$ *for each vertex* $v$*, then* $G$ *is Hamiltonian.*

**Exercise.** *If* $n \geqslant 3$*, then the clique* $K_n$ *is Hamiltonian.*

**Exercise.** *Let* $s, t \geqslant 2$*. The complete bipartite graph* $K_{s,t}$ *is Hamiltonian if and only if* $s = t$*.*

**Exercise** (The bound in Ore's condition is sharp). *For any* $n$*, show that there is a non-Hamiltonian graph* $G$ *on* $n$*-vertices such that* $\deg u + \deg v \geqslant n - 1$ *for each pair* $u$*,* $v$ *of nonadjacent vertices.*

**Theorem** (6.8, Bondy-Chvátal condition)**.** *Let* $G$ *be a graph on* $n \geqslant 3$ *vertices. Assume* $u$, $v$ *are nonadjacent vertices such that* $\deg u + \deg v \geqslant n$. *Then* $G$ *is Hamiltonian if and only if* $G + uv$ *is Hamiltonian.*

**Definition.** *Let* $G$ *be a graph on* $n$ *vertices. The closure* $C(G)$ *of* $G$ *is the graph obtained from* $G$ *by recursively adding edges* $uv$ *whenever* $u$, $v$ *are nonadjacent vertices such that* $\deg u + \deg v \geqslant n$, *until no such vertices exist.*

**Example.**

**Corollary** (6.9)**.** *A graph is Hamiltonian if and only if its closure is Hamiltonian.*

For the last theorem, let us denote

$$N_j(G) = |\{v \in V(G\colon \deg v \leqslant j)\}| = \text{ number of vertices with degree at most } j$$

**Theorem** (6.11, Pósa's condition). *Let* $G$ *be a graph on* $n \geqslant 3$ *vertices.*
*If for each integer* $j$ *with* $1 \leqslant j < n/2$ *we have*

$$N_j(G) < j$$

*then* $G$ *is Hamiltonian.*