Recall: Components in a graph G are equivalence classes under the relation on $V(G)$ defined by $u\mathcal{R}v$ iff there is a u–v path.

Definition. *Let G be a graph. We define a relation $\mathcal{B}$ on $E(G)$ by $e\mathcal{B}f$ iff $e = f$ or there is a cycle on G that uses e and f.*

Theorem (5.8)**.** *The relation $\mathcal{B}$ defined above is an equivalence relation.*

Last time:

- $v \in V(G)$ is a cut-vertex if $G - v$ has more components than G.
- We say that a connected graph G is non-separable if it contains no cut-vertices.
- G is non-separable iff every two vertices lie on a common cycle.

Definition. *A **block** of G is a maximal non-separable subgraph of G.*

Example. *Identify cut-vertices and blocks in the following graph*

Alternative definition of blocks:

- Last time: equivalence relation on $E(G)$: $e\mathcal{B}f$ iff $e = f$ or there is a cycle on G that uses e and f.
- Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ denote the equivalence classes under $\mathcal{B}$. Thus $E(G) = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_k$.
- Each equivalence class $\mathcal{E}$ induces a subgraph B with $E(B) = \mathcal{E}$ and $V(B) = \bigcup_{e \in \mathcal{E}} e$ (all vertices of G incident to some edge in $\mathcal{E}$.)

Theorem. *Each* B_i *is a block and all blocks of* G *arise in this way.*

Corollary (5.9)**.** *Every two distinct blocks* B_1 *and* B_2 *in a connected graph* G *satisfy:*

(a) B_1 *and* B_2 *are edge-disjoint.*

(b) B_1 *and* B_2 *have at most one common vertex.*

(c) If B_1 *and* B_2 *have a vertex* v *in common, then* v *is a cut-vertex of* G.

Exercise (5.6)**.** *Prove that a 3-regular graph* G *has a cut-vertex if and only if* G *has a bridge.*

Exercise (5.13)**.** *Assume* G *is a connected graph with cut-vertices. Prove or disprove: if* $u, v \in V(G)$ *are such that* $d(u, v) = \operatorname{diam} G$*, then no block of* G *contains* u *and* v*.*

Definition. *Let G be a graph*

- *A **vertex-cut** is a subset $U \subset V(G)$ such that $G - U$ is disconnected.*
- *The **vertex-connectivity** $\kappa(G)$ is the minimum cardinality of a vertex-cut. For $G = K_n$, we set $\kappa(G) = n-1$.*
- *If k is a non-negative integer, we say that G is k-connected if $\kappa(G) \geqslant k$.*

  Note: : Assume G has n vertices.

(1) $0 \leqslant \kappa(G) \leqslant n - 1$

(2) $\kappa(G) = 0$ iff G is disconnected

(3) $\kappa(G) = 1$ iff G has a cut-vertex

(4) G is 2-connected iff G is non-separable.

Example. *Compute $\kappa(G)$ for the following graphs.*

Definition. *Let G be a graph*

- *An **edge-cut** is a subset $F \subset E(G)$ such that $G - F$ is disconnected.*
- *The **edge-connectivity** $\lambda(G)$ is the minimum cardinality of an edge-cut. For $G = K_1$, we set $\lambda(G) = 0$.*
- *If k is a non-negative integer, we say that G is k-edge-connected if $\lambda(G) \geqslant k$.*

  Note: : Assume G has n vertices.

(1) $0 \leqslant \lambda(G) \leqslant n - 1$ (2) $\lambda(G) = 0$ iff G disconnected (3) $\lambda(G) = 1$ iff G has a bridge.

Example. *Compute $\lambda(G)$ for the following graphs.*