Last week:

- Some common graphs (complete bipartite, complete t-partite).
- Operations involving graphs (complement, disjoint union, join, cartesian product).
- Multigraphs and digraphs.
- Vertex degrees (handshaking lemma); interactions with connectivity.
- r-regular graphs (all vertices have degree r).
    - If r and n are not both odd and $r \leqslant n-1$, there exists an r-regular graph in n-vertices.
    - Any graph is contained in infinitely many regular graphs.

This week:

- Degree sequences.
- Graphical sequences, Havel–Hakimi theorem.
- Graph isomorphisms and automorphisms.

Definition. *A* ***degree sequence*** *of a graph in n-vertices is a list of its n vertex degrees (don't discard repeated degrees).*

- If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then a degree sequence is $\deg v_1, \deg v_2, \ldots, \deg v_n$ (depends on the labeling of $V(G)$).
- Usual choice: arrange vertex degrees in non-decreasing list $\deg v_1 \geqslant \deg v_2 \geqslant \cdots \geqslant \deg v_n$.

Example (Some graphs and their degree sequences)**.**

Next: which sequences come from some graph?

Definition. *A finite sequence* s *of nonnegative integers is called* ***graphical*** *if it is a degree sequence of some graph* G*. In this case we say that* G *is a realization of* s*.*

Exercise. *Suppose* $s\colon d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ *is graphical. Then*

- s *has an even number of odd entries.*

- $n-1 \geqslant d_1 \geqslant \cdots \geqslant d_n$.

- *if* $d_1, \ldots, d_k = n-1$*, then* $d_n \geqslant k$.

Example. *Which of the following sequences are graphical?*

(a) $s_1\colon 3, 3, 2, 2, 1, 1.$

(b) $s_2\colon 6, 5, 5, 4, 3, 3, 3, 2, 2.$

(c) $s_3\colon 7, 6, 4, 4, 3, 3, 3.$

(d) $s_4\colon 3, 3, 3, 1.$

Theorem ((2.10) Havel–Hakimi). *A sequence* $s\colon d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$*, where* $d_1 \geqslant 1$*, is graphical if and only if the sequence*

$$s_1\colon d_2 - 1, d_3 - 1, \ldots, d_{d_1+1} - 1, d_{d_1+2}, \ldots, d_n$$

is graphical.

Example (2.11). *Decide whether* $s\colon 5,4,3,3,2,2,2,1,1,1$ *is graphical.*

Theorem (Erdös-Gallai (not in the book)). *A sequence* $s\colon d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ *is graphical if and only if*

(1) $\sum_{i=1}^{n} d_i$ *is even, and* (2) *for all* $k \in \{1, \dots, n\}$, $\sum_{i=1}^{k} d_i \leqslant k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$.

Proof. (Only $\implies$) Assume $G = (V, E)$ is a realization of s with vertices $V = \{v_1, \dots, v_n\}$ such that $d_i = \deg v_i$.

Step 0: $\sum_{i=1}^{n} d_i$ is even:

Now we prove (2) in a series of steps. Fix k and consider $\Omega = \{(x, y) \in \{v_1, \dots, v_k\} \times V\colon xy \in E(G)\}$.

Step 1: $|\Omega| = \sum_{i=1}^{k} d_i$:

Step 2: We have a partition $\Omega = \Omega_1 \sqcup \Omega_2$, where

$$\Omega_1 = \{(x, y) \in \{v_1, \dots, v_k\} \times \{v_1, \dots, v_k\}\colon xy \in E(G)\}, \qquad \Omega_2 = \{(x, y) \in \{v_1, \dots, v_k\} \times \{v_{k+1}, \dots, v_n\}\colon xy \in E(G)\}.$$

Step 3: $|\Omega_1| \leqslant k(k-1)$

Step 4: $|\Omega_2| \leqslant \sum_{i=k+1}^{n} \min\{k, d_i\}$.

Step 5: Finish the proof of (2).

□