Pre-Nichols algebras with finite Gelfand-Kirillov dimension

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Joint with Andruskiewitsch (arXiv:2002.11087) & Angiono-Campagnolo (arXiv:2009.04863).

University Quantum Symmetries Lectures.

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Contents

Motivation

Pointed Hopf algebras: coradical and infinitesimal braiding

Fix field $\Bbbk = \overline{\Bbbk}$ with $\operatorname{char} \Bbbk = 0$. Let A Hopf algebra.

- Coradical of A is $A_0 :=$ sum of simple subcoalgebras.
- A is pointed if $A_0 = \Bbbk G(A)$, the group algebra of group-likes.

Example: any Hopf algebra generated by group-likes and skew-primitives.

$\rightsquigarrow$ A_0 extends to the coradical filtratrion $A = \bigcup_{n\geq 0} A_n$, Hopf algebra filtration.

$\rightsquigarrow$ Associated $\operatorname{gr} A$ is Hopf alg with $A_0 \hookrightarrow \operatorname{gr} A$ and $\operatorname{gr} A \twoheadrightarrow A_0$ composing to id.

Radford-Majid correspondence:

$$\left\{K \xrightarrow{\iota} H \xrightarrow{\pi} K \text{ with } \pi\iota = \mathrm{id}_K\right\} \longleftrightarrow \left\{\text{Hopf algs in } {}^K_K\mathcal{YD}\right\}, \quad H \mapsto H^{\mathrm{co}\,\pi}, \quad R\#K \leftarrow R$$

$\rightsquigarrow$ Recover $\operatorname{gr} A \simeq R\#A_0$ with R graded Hopf algebra in ${}^{A_0}_{A_0}\mathcal{YD}$.

- $R = \bigoplus_{n\geq 0} R^n$ is the diagram and R^1 is the infinitesimal braiding of A.

Next: What type of objects are these diagrams?

Nichols algebras

Fix a group Γ.

- Category ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$: objects $V = \bigoplus_{g\in\Gamma} V_g$ with Γ-action $h \cdot V_g = V_{hgh^{-1}}$.
  Braiding $V \otimes W \xrightarrow{c} W \otimes V$, $c(v \otimes w) = g \cdot w \otimes v$ for $v \in V_g$.
- $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD} \rightsquigarrow T(V)$ graded Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ (declare $V \subset$ primitives).

  Fact: exists unique maximal graded Hopf ideal $\mathcal{J}(V) \subset V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots$

  Definition: $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ is the Nichols algebra of V.
- A post-Nichols algebra of V is graded Hopf algebra $\mathcal{E} = \bigoplus_{n\geq 0} \mathcal{E}^n$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with $\mathcal{E}^0 = \Bbbk$, $\mathcal{E}^1 = V$ such that the original grading is the coradical one.

  Fact: id_V extends to inclusions $\mathcal{B}(V) \hookrightarrow \mathcal{E} \hookrightarrow T^c(V)$ of Hopf algebras in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

  Example: the diagram of a Hopf alg. with coradical $\Bbbk\Gamma$ and inf. braiding V.

Generation in degree 1 Conjecture, Andruskiewitsch-Schneider '00

If $\dim \mathcal{B}(V) < \infty$ the unique finite dimensional post-Nichols of V is $\mathcal{B}(V)$.

- If Γ is abelian the conjecture holds (Angiono).

Nichols algebras of diagonal type

From now on $\Gamma = \mathbb{Z}^\theta = g_1\mathbb{Z} \oplus \cdots \oplus g_\theta\mathbb{Z}$.

- Given $\mathfrak{q} = (q_{ij}) \in \mathrm{Mat}_\theta(\Bbbk^\times)$ define $V^\mathfrak{q} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by

  $V^\mathfrak{q}$ has basis $\{x_1, \dots, x_\theta\}$, coaction $x_i \mapsto g_i \otimes x_i$, action $g_i \cdot x_j = q_{ij}x_j$.

Thus the braiding is $c^\mathfrak{q}(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$. These are called of diagonal type.

- Dynkin diagram of $\mathfrak{q}$: is the *decorated* graph with

· vertices $\{1, \dots, \theta\}$, vertex i labeled with q_{ii};

· edge between i & j iff $\widetilde{q}_{ij} := q_{ij}q_{ji} \neq 1$. Such edged is labeled with $\widetilde{q}_{ij}$.

Example 1: $\widetilde{q}_{ij} = 1$ for all $i \neq j$, i.e. Dynkin diagram is totally disconnected.

$\rightsquigarrow \mathcal{B}(V^\mathfrak{q}) = T(V^\mathfrak{q})/\langle x_i^{\mathrm{ord}\, q_{ii}},\ x_ix_j - q_{ij}x_jx_i :\ i < j\rangle$ quantum linear space.

Nichols algebras of Cartan type

Example 2: $\mathfrak{q}$ is of Cartan type if exists GCM $\mathbf{a} = (a_{ij})$ such that $\widetilde{q}_{ij} = q_{ii}^{a_{ij}}$.
After normalization $-\operatorname{ord} q_{ii} < a_{ij} \leq 0$ we say $\mathfrak{q}$ is of Cartan type $\mathbf{a}$.

Theorem: (Andruskiewitsch-Schneider, Heckenberger) Assume $\mathbf{a}$ indecomp.
Then $\dim \mathcal{B}(V^{\mathfrak{q}}) < \infty$ iff $\mathbf{a}$ is finite and $\operatorname{ord} q_{ii} < \infty$.

Example 2,5: Let $\mathbf{a}$ finite Cartan matrix, $d_1, \dots, d_\theta$ symmetrizing $\mathbf{a}$.

Let q with usual restrictions on $\operatorname{ord} q < \infty$. Let $q_{ij} = q^{d_i a_{ij}}$.

Then $\mathcal{B}(V^{\mathfrak{q}}) \simeq \mathrm{u}_q^+(\mathfrak{g})$.

Lusztig's divided powers $\mathrm{U}_q^+(\mathfrak{g}) \hookleftarrow \mathrm{u}_q^+(\mathfrak{g})$ is post-Nichols with finite GKdim.

Punchline: Nichols algebras of diagonal type with finite dim have interesting post-Nichols with finite GKdim.

A Nichols algebra of super type

Example 3: Let q of odd order and $\mathfrak{q}$ with Dynkin diagram

$$\overset{q^{-2}}{\underset{1}{\circ}} \xrightarrow{\;q^{2}\;} \overset{q^{-2}}{\underset{2}{\circ}} \cdots\cdots \overset{-1}{\underset{m}{\circ}} \cdots\cdots \overset{q^{2}}{\underset{n-1}{\circ}} \xrightarrow{\;q^{-2}\;} \overset{q^{2}}{\underset{n}{\circ}} .$$

Then $\mathcal{B}(V^{\mathfrak{q}}) \simeq \mathrm{u}_q^+(\mathfrak{sl}(m|n))$.

We have $\mathrm{U}_q^+(\mathfrak{sl}(m|n)) \hookleftarrow \mathrm{u}_q^+(\mathfrak{sl}(m|n))$ post-Nichols with finite GKdim.

Can be generalized to other super quantum groups.

Punchline: Nichols algebras of diagonal type with finite dim give a general framework.

Our problem

Goal: classify Hopf alg with finite GKdim and diagonal infinitesimal braiding.

(A) classify (fin. dim.) $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $\operatorname{GKdim} \mathcal{B}(V) < \infty$,

(B) for such V classify all post-Nichols with finite GKdim,

which reduces to (Andruskiewitsch, Angiono, Heckenberger)

(C) classify pre-Nichols algebras of V with finite GKdim.

• A pre-Nichols algebra of V is a graded Hopf algebra $\mathcal{B} = \bigoplus_{n\geq 0} \mathcal{B}^n$ in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ with $\mathcal{B}^0 = \Bbbk$, $\mathcal{B}^1 = V$ which is generated by $\mathcal{B}^1$.

Fact: id_V extends to projections $T(V) \twoheadrightarrow \mathcal{B} \twoheadrightarrow \mathcal{B}(V)$ of Hopf algebras in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

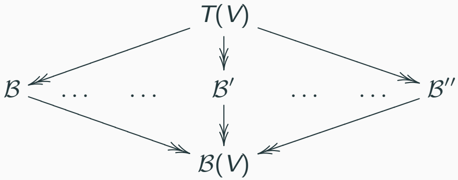
We focus on problem (C) for two families $V^{\mathfrak{q}}$ of diagonal type:

(1) $\mathfrak{q}$ with totally disconnected Dynkin diagram,

(2) $\mathfrak{q}$ with connected Dynkin diagram and $\dim(\mathcal{B}(V^{\mathfrak{q}})) < \infty$.

Eminent pre-Nichols

Poset $\mathfrak{Pre}(V)$ of all pre-Nichols of V:



$\rightsquigarrow$ minimal

$\rightsquigarrow$ maximal

$\mathfrak{Pre}_{\mathrm{fGK}}(V) :=$ subposet of those with finite GKdim.

Definition (Andruskiewitsch-S)

A pre-Nichols $\widehat{\mathcal{B}}$ is eminent if it is the minimum of $\mathfrak{Pre}_{\mathrm{fGK}}(V)$.

This is: $\operatorname{GKdim} \widehat{\mathcal{B}} < \infty$ and for each $\mathcal{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$ there is a projection $\widehat{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ of braided Hopf algebras that is the identity on V.

Today: decide if exist, and in such case find, eminent pre-Nichols for our braidings

Totally disconnected diagrams

Terminology

- Recall $V^{\mathfrak{q}}$ has basis $\{x_1, \dots, x_\theta\}$ and braiding $c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i$.
- Disconnected condition: $\quad q_{ij} q_{ji} = 1$ for $i \neq j$.
- Nichols algebra: $\quad \mathcal{B}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}}) / \langle x_i^{\operatorname{ord} q_{ii}},\ x_i x_j - q_{ij} x_j x_i : \ i < j \rangle$.
  PBW basis $\{x_\theta^{n_\theta} \cdots x_1^{n_1} : 0 \leq n_i < \operatorname{ord} q_{ii}\}$.
- Distinguished pre-Nichols: $\quad \widetilde{\mathcal{B}}_{\mathfrak{q}} := T(V) / \langle x_i x_j - q_{ij} x_j x_i : \ i < j \rangle$.
  PBW basis $\{x_\theta^{n_\theta} \cdots x_1^{n_1} : 0 \leq n_i\}$ and $\operatorname{GKdim} = \theta$.

Partition $\{1, \dots, \theta\} = \mathbb{I}^{<3} \sqcup \mathbb{I}^{3} \sqcup \mathbb{I}^{>3} \sqcup \mathbb{I}^{\infty}$ where

$$\mathbb{I}^{\infty} = \{i\colon \operatorname{ord} q_{ii} = \infty\}, \qquad \mathbb{I}^{N} = \{i\colon \operatorname{ord} q_{ii} = N\},\ N \geq 1,$$

$$\mathbb{I}^{>3} = \bigcup_{N>3} \mathbb{I}^{N}, \qquad \mathbb{I}^{<3} = \{i\colon q_{ii} = \pm 1\}.$$

For $\star \in \{<3, 3, >3, \infty\}$ put $V^{\star} = \Bbbk\{x_i\colon i \in \mathbb{I}^{\star}\} \subset V$, so

$$V = V^{<3} \oplus V^{3} \oplus V^{>3} \oplus V^{\infty}.$$

Example: super symmetric algebras

Super symmetric condition: $V = V^1 \oplus V^2$, i.e. $q_{ii}^2 = 1$.

Lemma (Andruskiewitsch-S)

If $\mathfrak{q}$ y $\mathfrak{p}$ have same Dynkin diagram there is a posets iso

$$\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{q}) \simeq \mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{p}) .$$

$\rightsquigarrow$ Consider instead $(V^{\mathfrak{p}}, c^{\mathfrak{p}})$ with $p_{ij} = \pm 1$.

- $\mathfrak{Pre}(\mathfrak{p})$ = super enveloping algebras $U(\mathfrak{n})$ of $\mathfrak{n} = \oplus_{j\geq 1}\mathfrak{n}^j$ graded Lie super algebra generated by $\mathfrak{n}^1 = V$.
- $\mathfrak{Pre}_{\mathrm{fGK}}(\mathfrak{p}) = \{U(\mathfrak{n}) \colon \dim \mathfrak{n} < \infty\}$. But GKdim $U(\mathfrak{n}) = \dim \mathfrak{n}$ grows arbitrary.

Eminent pre-Nichols might not exist!

Main result for totally disconnected $\mathfrak{q}$

Recall $V = V^{<3} \oplus V^{3} \oplus V^{>3} \oplus V^{\infty}$.

Theorem (Andruskiewitsch-S)

(1) For $\star \in \{3, >3, \infty\}$ the distinguished pre-Nichols $\widetilde{\mathcal{B}}(V^{\star})$ is eminent.

(2) All $\mathcal{B} \in \mathfrak{Pre}_{\mathrm{fGK}}(V)$ split as $\mathcal{B} \simeq \mathcal{B}^{\leq 3} \underline{\otimes} \mathcal{B}^{>3} \underline{\otimes} \mathcal{B}^{\infty}$ where $\mathcal{B}^{\star} \subset \mathcal{B}$ is the subalgebra generated by $V^{\star}$.

(3) Let $V = \Bbbk x_1 \oplus \Bbbk x_2$ with $x_1 \in V^{3}$ and $x_2 \in V^{1}$. Then

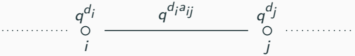
$$\breve{\mathcal{B}}(V) = T(V)/\langle (\operatorname{ad}_c x_1)^4 x_2,\ (\operatorname{ad}_c x_2)^2 x_1 \rangle$$

is an eminent pre-Nichols with $\operatorname{GKdim} = 6$.

Moreover $\breve{\mathcal{B}}(V)$ has a PBW basis that looks like that of G_2 ...

(Non genuine) quantum Serre relations

- Let $\mathbf{a} = (a_{ij})$ indecomp. GCM with $d_1, \dots, d_\theta$ symmetrizing integers.
- Let $q \in \Bbbk^\times$ and $\mathfrak{q}$ with diagram



Beware: this is Cartan type but not necessarily type $\mathbf{a}$.

- Define $\check{\mathcal{B}}_{\mathfrak{q}} = T(V^{\mathfrak{q}}) / \langle (\mathrm{ad}_c\, x_i)^{1-a_{ij}} x_j \colon i \neq j \rangle$.

Proposition (Andruskiewitsch-S)

GKdim $\check{\mathcal{B}}_{\mathfrak{q}} \geq \dim \mathfrak{g}^+$ where $\mathfrak{g}(\mathbf{a}) = \mathfrak{g}^+ \oplus \mathfrak{h} \oplus \mathfrak{g}^-$ associated Kac-Moody.

Notation: $T(V^{\mathfrak{q}})$ has adjoint representation $\mathrm{ad}_c \colon T(V^{\mathfrak{q}}) \to \mathrm{End}\, T(V^{\mathfrak{q}})$.

For $x \in V \rightsquigarrow (\mathrm{ad}_c\, x) y = m(\mathrm{id} - c)(x \otimes y)$.

Denote $x_{i_1 \dots i_k} := (\mathrm{ad}_c\, x_{i_1}) x_{i_2 \dots i_k} = x_{i_1} x_{i_2 \dots i_k} - q_{i_1 i_2} \cdots q_{i_1 i_k} x_{i_2 \dots i_k} x_{i_1}$.

Finite connected diagrams

Terminology: root system and Weyl grupoid

Let $\theta \geq 2$, canonical basis $\{\alpha_1, \dots, \alpha_\theta\}$ of $\mathbb{Z}^\theta$ and $\mathfrak{q} \in \mathrm{Mat}_\theta(\Bbbk^\times)$.

- Roots of $\mathfrak{q}$: set $\Delta_+^{\mathfrak{q}} \subset \mathbb{Z}^\theta$ degrees of monomials of a PBW basis for $\mathcal{B}(V^{\mathfrak{q}})$.
- GCM: $C^{\mathfrak{q}} \in \mathrm{Mat}_\theta(\mathbb{Z})$ with $c_{ij}^{\mathfrak{q}} = -\min\{n \geq 0 \colon (n+1)_{q_{ii}}(1 - q_{ii}^n \widetilde{q}_{ij}) = 0\}$.
- Reflections: $s_i^{\mathfrak{q}} \in GL(\mathbb{Z}^\theta)$ given by $s_i^{\mathfrak{q}}(\alpha_j) = \alpha_j - c_{ij}^{\mathfrak{q}}\alpha_i$.
- New braiding: $\rho_i(\mathfrak{q}) \in \mathrm{Mat}_\theta(\Bbbk^\times)$ given by $(\rho_i(\mathfrak{q}))_{jk} = \mathfrak{q}(s_i^{\mathfrak{q}}(\alpha_j), s_i^{\mathfrak{q}}(\alpha_k))$.
- Lusztig's isomorphisms $s_i^{\mathfrak{q}} : \mathcal{U}_{\mathfrak{q}} \to \mathcal{U}_{\rho_i(\mathfrak{q})}$ of algebras.
- Weyl class of $\mathfrak{q}$: $\mathcal{X} = \{\rho_{j_1} \dots \rho_{j_n}(\mathfrak{q}) : 1 \leq j_1, \dots, j_n \leq \theta\}$.
- Root system of $\mathfrak{q}$: is the fibration $(\Delta^{\mathfrak{p}})_{\mathfrak{p} \in \mathcal{X}}$.
- Weyl groupoid $\mathcal{W}$ of $\mathfrak{q}$: $\mathrm{span}\{s_i^{\mathfrak{p}} : \mathfrak{p} \in \mathcal{X}\}$ in $\mathcal{X} \times GL(\mathbb{Z}^\theta) \times \mathcal{X}$.
- If $|\Delta_+^{\mathfrak{q}}| < \infty$ let $\omega_0^{\mathfrak{q}} = \sigma_{i_1}^{\mathfrak{q}} \cdots \sigma_{i_\ell}$ reduced expression of maximal length in $\mathcal{W}$. Then $\{x_{\beta_1}^{n_1} \cdots x_{\beta_\ell}^{n_\ell} : 0 \leq n_k < N_{\beta_k}\}$ with $\beta_k = s_{i_1}^{\mathfrak{q}} \cdots s_{i_{k-1}}(\alpha_{i_k})$ is a basis of $\mathcal{B}(V^{\mathfrak{q}})$.

Finite dim **vs finite** GKdim

Theorem [H09] [1]

Classification of connected $\mathfrak{q}$'s with finite root system.

- All such $\mathfrak{q}$ have $\operatorname{GKdim} \mathcal{B}(V^{\mathfrak{q}}) < \infty$.
- They either depend on some fixed roots of 1 (in which case $\dim \mathcal{B}(V^{\mathfrak{q}}) < \infty$) or on generic parameters (which for some specialization $\dim \mathcal{B}(V^{\mathfrak{q}}) < \infty$).
- This contains all connected $\mathfrak{q}$'s with $\dim \mathcal{B}(V^{\mathfrak{q}}) < \infty$.
- Andruskiewitsch-Angiono: Lie-theoretical organization:

· Cartan,	· standar,	· super,
· modular,	· supermodular,	· UFO.

- Going back to problem (A):

Conjecture [AAH] [2]

If $\operatorname{GKdim} \mathcal{B}(V^{\mathfrak{q}}) < \infty$, the root system of $\mathfrak{q}$ is finite.

[1] [H09] I. Heckenberger, *Classification of arithmetic root systems*. Adv. Math. 2009
[2] [AAH] *On finite GKdim Nichols algebras over abelian groups*. Mem. Amer. Math. Soc.

Distinguished pre-Nichols algebras

For connected $\mathfrak{q}$ with $\dim \mathcal{B}(V^{\mathfrak{q}}) < \infty$ Angiono [3] defines the Distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$. Some features:

- If $\mathcal{B}(V^{\mathfrak{q}}) = \mathrm{u}_q^+(\mathfrak{g})$, then $\widetilde{\mathcal{B}}(V^{\mathfrak{q}}) = \mathrm{U}_q^+(\mathfrak{g})$.
- Kernel of $\widetilde{\mathcal{B}}(V^{\mathfrak{q}}) \twoheadrightarrow \mathcal{B}(V^{\mathfrak{q}})$ is central subalgebra generated by some powers $x_\beta^{N_\beta}$.
- $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$ inherits Lusztig's isomorphisms from $\mathcal{B}(V^{\mathfrak{q}})$.
- $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$ has a PBW basis with the same monomials of $\mathcal{B}(V^{\mathfrak{q}})$ ($\neq$ heights).
- $\operatorname{GKdim} \widetilde{\mathcal{B}}(V^{\mathfrak{q}}) < \infty$.

Punchline: these $\mathfrak{q}$ have interesting pre-Nichols algebras with finite GKdim.

Angiono: is $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ eminent?

Next: we answer this for types Cartan, standar and super. When it fails, we find substitutes in all but two cases.

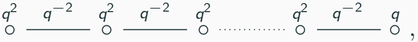
[3] [An] I. Angiono, *Distinguished pre-Nichols algebras*. Transf. Groups (2016).

Dynkin diagrams of finite Cartan type

They depend on a root of unity q:

A_θ : 

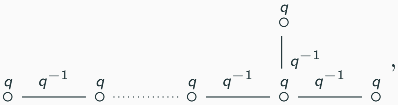
$\theta \geq 2,\ \mathrm{ord}\, q \geq 2;$

B_θ : 

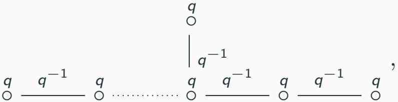
$\theta \geq 2,\ \mathrm{ord}\, q \geq 3;$

C_θ : 

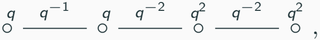
$\theta \geq 3,\ \mathrm{ord}\, q \geq 3;$

D_θ : 

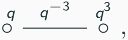
$\theta \geq 4,\ \mathrm{ord}\, q \geq 2;$

E_θ : 

$6 \leq \theta \leq 8,\ \mathrm{ord}\, q \geq 2;$

F_4 : 

$\mathrm{ord}\, q \geq 3;$

G_2 : 

$\mathrm{ord}\, q \geq 4.$

Main result for connected diagrams

Theorem (Andruskiewitsch-S, Angiono-Campagnolo-S)

The distinguished pre-Nichols algebra is eminent in the following cases:

(I) $\mathfrak{q}$ is Cartan different from

$$A_2 : \quad \overset{\omega}{\circ} \overset{\omega^{-1}}{\rule[3pt]{40pt}{0.4pt}} \overset{\omega}{\circ}, \quad \operatorname{ord}\omega = 3;$$

$$A_\theta : \quad \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ}, \qquad \theta \geq 2;$$

$$D_\theta : \quad \begin{array}{c} \hphantom{\overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}}} \overset{-1}{\circ} \hphantom{\overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ}} \\ \hphantom{\overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}}} \Big|{\scriptstyle -1} \hphantom{\overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ}} \\ \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \cdots\cdots \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \overset{-1}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ}, \end{array} \qquad \theta \geq 4.$$

(II) $\mathfrak{q}$ is standard.

(III) $\mathfrak{q}$ is super different from

$$\mathbf{A}_3(q|\{2\}) : \quad \overset{q^{-1}}{\circ} \overset{q}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \overset{q^{-1}}{\rule[3pt]{40pt}{0.4pt}} \overset{q}{\circ}, \qquad 2 < \operatorname{ord} q < \infty;$$

$$\mathbf{A}_3(q|\{1,2,3\}) : \quad \overset{-1}{\circ} \overset{q^{-1}}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ} \overset{q}{\rule[3pt]{40pt}{0.4pt}} \overset{-1}{\circ}, \qquad 2 < \operatorname{ord} q < \infty.$$

About the proof

- The Nichols algebra $\mathcal{B}(V^{\mathfrak{q}})$ has a presentation by relations (Angiono).
E.g. $x_\beta^{N_\beta} = 0$, $(\operatorname{ad}_c x_i)^{1-c_{ij}} x_j = 0$, and more ...

- The distinguished pre-Nichols $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$ is defined by same relations but not vanishing powers $x_\beta^{N_\beta}$ when β is a Cartan root.

- Any pre-Nichols $\mathcal{B}$ is a quotient $T(V^{\mathfrak{q}}) \twoheadrightarrow \mathcal{B} \twoheadrightarrow \mathcal{B}(V^{\mathfrak{q}})$. Goal: show that each defining relation r of $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$ vanishes under $T(V^{\mathfrak{q}}) \twoheadrightarrow \mathcal{B}$.

- We show that if $r \neq 0$ in $\mathcal{B}$ we get a Dynkin diagram with infinite root system inside $\mathcal{P}(\mathcal{B}) =$ primitives.

- Hence $r \neq 0$ in $\mathcal{B}$ implies $\operatorname{GKdim} \mathcal{B} = \infty$.

About the exceptions

- Why? Certain defining rels r of $\widetilde{\mathcal{B}}(V^{\mathfrak{q}})$ give finite diagrams in $\mathcal{P}(\mathcal{B})$.
- Guiding idea: can $(\operatorname{ad}_c x_i) r \neq 0$ in $\mathcal{B}$?

Example: Cartan A_2, $\overset{\omega}{\circ} \overset{\omega^{-1}}{\text{------}} \overset{\omega}{\circ}$ with $\operatorname{ord} \omega = 3$.

Here $\mathcal{B}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}})/\langle x_1^3, x_2^3, x_{12}^3, x_{112}, x_{221}\rangle$ and $\widetilde{\mathcal{B}}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}})/\langle x_{112}, x_{221}\rangle$.

Consider $\widehat{\mathcal{B}}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}})/\langle x_{1112}, x_{2112}, x_{1221}, x_{2221}\rangle$.

Define $\widehat{\mathcal{Z}}_{\mathfrak{q}}$ = subalgebra of $\widehat{\mathcal{B}}(V^{\mathfrak{q}})$ gen by x_1^3, x_2^3, x_{12}^3, x_{112}, x_{221}.

Theorem (Andruskiewitsch-S)

(1) $\widehat{\mathcal{Z}}_{\mathfrak{q}}$ is a Hopf subalgebra and $\left(\operatorname{ad}_c \widehat{\mathcal{B}}(V^{\mathfrak{q}})\right) \widehat{\mathcal{Z}}_{\mathfrak{q}} = 0$.

(2) $\widehat{\mathcal{Z}}_{\mathfrak{q}}$ is a q-polynomial algebra in five variables.

(3) We have an extension of braided Hopf algebras $\widehat{\mathcal{Z}}_{\mathfrak{q}} \hookrightarrow \widehat{\mathcal{B}}(V^{\mathfrak{q}}) \twoheadrightarrow \mathcal{B}(V^{\mathfrak{q}})$.

(4) The pre-Nichols $\widehat{\mathcal{B}}(V^{\mathfrak{q}})$ is eminent with $\operatorname{GKdim} = 5$.

Other new eminents

Theorem (Angiono-Campagnolo-S)

(1) If $\mathfrak{q}$ is $\overset{q^{-1}}{\circ} \xrightarrow{\ q\ } \overset{-1}{\circ} \xrightarrow{\ q^{-1}\ } \overset{q}{\circ}$ with $2 < \operatorname{ord} q < \infty$, then

$$\widehat{\mathcal{B}}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}})/\langle x_2^2, x_{13}, x_{112}, x_{332}\rangle$$

is eminent pre-Nichols with GKdim $= 3$.

(2) If $\mathfrak{q}$ is $\overset{-1}{\circ} \xrightarrow{\ q^{-1}\ } \overset{-1}{\circ} \xrightarrow{\ q\ } \overset{-1}{\circ}$ with $2 < \operatorname{ord} q < \infty$, then

$$\widehat{\mathcal{B}}(V^{\mathfrak{q}}) = T(V^{\mathfrak{q}})/\langle x_1^2, x_2^2, x_3^2, x_{213}, [x_{123}, x_2]_c\rangle$$

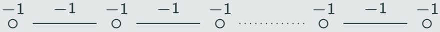
is eminent pre-Nichols with GKdim $= 3$ and basis

$$\big\{x_3^a x_{23}^b x_2^c x_{13}^d x_{123}^e x_{12}^f x_1^g :\ a, c, e, g \in \{0, 1\},\ b, d, f \geq 0\big\}.$$

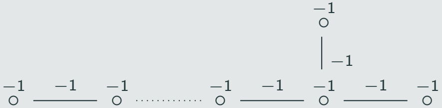
Summary on connected diagrams

Theorem (Andruskiewitsch-S, Angiono-Campagnolo-S)

Braidings of Cartan, standar and super type admit eminent pre-Nichols except in types

A_θ : 

, $\theta \geq 2$;

D_θ : 

, $\theta \geq 4$.

- In type D_θ we have a candidate (remains GKdim $< \infty$).
- For A_θ with $\theta > 2$ it seems one can cover $\mathfrak{Pre}_{\mathrm{fGK}}$ with **two** pre-Nichols.
- A_2?
- Work in progress: modular, supermodular and UFO families.
- Why are eminent pre-Nichols braided central extensions of the Nichols?

Thank you!