Last week:

- Planar graphs
- Planarity and chromatic numbers

This week:

- Ramsey numbers

Motivation:

- There is a party with N random guests.
- Want to ensure at least 5 know each other **or** that at least 5 are strangers.

What should N be? What is the smallest possible value for N?

Definition. *Fix an integer* $N \geqslant 2$ *and a 2-edge coloring* $f\colon E(K_N) \to \{$*red, blue*$\}$ *for the clique* K_N.
For a given graph G *we say that:*

- *The coloring* ***contains a red*** G *if* K_N *contains a copy of* G *whose edges are red.*
- *The coloring* ***contains a blue*** G *if* K_N *contains a copy of* G *whose edges are blue.*
- *The coloring* ***contains a monochromatic*** G *if it contains either a red or a blue* G.

Example.

Note: The motivating problem translates to:

Ramsey theory: what monochromatic structures **must** appear in **any** red-blue coloring of $E(K_N)$.

Definition. *For positive integers* m*,* n*, the* ***Ramsey number*** $R(m,n)$ *is the smallest integer* N *such that every red,blue-coloring of* $E(K_N)$ *contains either a red copy of* K_m *or a blue copy of* K_n*.*

Note 1: A red,blue-coloring of $E(K_N)$ corresponds to an N-vertex graph G and its complement $\overline{G}$.

Note 2: $R(m,n)$ is the smallest integer N such that for every N-vertex graph G, either G contains a copy of K_m or $\overline{G}$ contains a copy of K_n.

Note 3: $R(m,n)$ is the smallest integer N such that for every N-vertex graph G, either $\omega(G) \geqslant m$ or $\alpha(G) \geqslant n$.

Notes:

(1) A priori, it is not clear that $R(m, n)$ exists for all values of m, n.

(2) If $N = R(m, n)$ exists and M is any integer with $M \geqslant N$, then also every red,blue-coloring of $E(K_M)$ contains either a red copy of K_m or a blue copy of K_n.

(3) $R(m, n) = R(n, m)$.

(4) $R(1, n) = 1$ for every n.

(5) $R(2, n) = n$ for every n.

(6) If $m_1 \leqslant m_2$ and $n_1 \leqslant n_2$, then $R(m_1, n_1) \leqslant R(m_2, n_2)$.

Keep in mind during proofs: Given an integer N,

(1) In order to prove that $R(m, n) \geqslant N$, we must **construct a** red,blue-coloring of $E(K_{N-1})$ which has neither a red copy of K_m nor a blue copy of K_n.

(2) In order to prove that $R(m, n) \leqslant N$, we must **show that every** red,blue- coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n.

Theorem. $R(3, 3) = 6$.

Theorem. *For integers* $m, n \geqslant 2$, *if* $R(m-1, n)$ *and* $R(m, n-1)$ *both exist, then* $R(m, n)$ *also exists and*

$$R(m, n) \leqslant R(m-1, n) + R(m, n-1).$$

Corollary. *For integers* m, n*, the Ramsey number* $R(m, n)$ *exists and*

$$R(m,n) \leqslant \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}.$$

Corollary. *For any positive integer* n*, we have*

$$R(n,n) \leqslant \binom{2n-2}{n-1} \leqslant 2^{2n-2} \leqslant 4^n.$$

Theorem. $R(4,3) = 9$.

Theorem. $R(4,4) = 18$.

Best available bounds:

$$43 \leqslant R(5,5) \leqslant 48, \qquad 102 \leqslant R(6,6) \leqslant 165.$$