Last week:

- Degree sequences.
- Graphical sequences, Havel–Hakimi theorem, Erdös–Gallai (one direction).
- Adjacency matrix (how to store a graph).
- Graph isomorphisms and automorphisms.

This week:

- Trees.
- More trees.
- Spanning trees.

Definition. *An edge* e *of a graph* G *is a* ***bridge*** *if* $G - e$ *has more components than* G*.*

- In case G is connected, e is a bridge if and only if $G - e$ is disconnected.

Example. *Identify edges in the following graph.*

Exercise. *(a) If* $e = uv$ *is a bridge in* G*, then* u*,* v *lie in different components of* $G - e$*.*

(b) If $e = uv$ *is a bridge in* G*, then there is a unique* u*–*v *path in* G*.*

Theorem (4.1). *An edge* e *of a graph* G *is a bridge if and only if* e *lies on no cycle of* G*.*

Definition. *A **tree** is a connected graph with no cycles.*

Example (All trees on 4 vertices)**.**

- A **forest** is a graph with no cycles.
- A tree is a star if it has exactly one vertex that is not a leaf.
- A tree is a **double star** if it has exactly two vertices that are not leaves.
- A tree on at least 3 vertices is a **caterpillar** if removing all leaves results in a path. That path is the **spine**.

Example (Forests, (double) stars, caterpillars)**.**

Theorem (Not in the book)**.** *Let* G *be a graph. If there are* $x, y \in V(G)$ *connected by at least two paths, then* G *contains a cycle.*

Proof. Consider two different x–y paths $(x = u_0, u_1, \dots, u_k = y)$ and $(x = v_0, v_1, \dots, v_\ell = y)$, say with $k \leqslant \ell$.

Step 1: Let i denote the largest index for which $u_j = v_j$ for all $j \in \{0, \dots, i\}$. Then $i < k$.

Step 2: There is a smallest $s \in \{i+1, \dots, k\}$ for which $u_s \in \{v_{i+1}, \dots, v_\ell\}$.

Step 3: Let $t \in \{i+1, \dots, \ell\}$ such that $u_s = v_t$. Then $s \neq i+1$ or $t \neq i+1$.

Step 4: $(v_i = u_i, u_{i+1}, \dots, u_s = v_t, v_{t-1}, \dots v_{i+1})$ is a cycle in G. □

Theorem (4.2). *A graph* G *is a tree if and only if every two vertices of* G *are connected by a unique path.*

Theorem (4.3). *Every nontrivial tree has at least two leaves.*

Note: If T is a tree on n vertices and v is a leaf, then $T - v$ is a tree with $n - 1$ vertices.

Theorem (4.4). *If* T *is a tree, then* $|E(T)| = |V(G)| - 1$. *(A tree on* n *vertices has exactly* $n - 1$ *edges.)*

Corollary (4.6). *If a forest has exactly n vertices and k components, then it has $n - k$ edges.*

Exercise. *Show that every tree is bipartite.*

Exercise. *Prove that a graph G is a tree if and only if G contains no cycle but $G + uv$ does contain a cycle for each pair of non-adjacent vertices u, v in G.*

Exercise. *Let T be a tree. For each $i \geqslant 1$, let n_i denote the number of vertices of degree i. Show that*

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \dots$$