Last week:

- Walks, trails and paths.
- Connected graphs (any two vertices are connected by a path).
  - Distance and geodesics between vertices.
  - Diameter of a graph (greatest distance between two vertices).
- Some common graphs (Paths P_n, Cycles C_n, Cliques K_n).
- Independent set of vertices (induced graph is edgeless).
- Bipartite graphs (the set of vertices can be partitioned with two independent sets).

This week:

- Some common graphs.
- Operations involving graphs (complement, disjoint union, join, cartesian product).
- Multigraphs and digraphs.
- Vertex degrees (handshaking lemma).
- Regular graphs.

Definition. *A bipartite graph* G *with bipartition* $V(G) = X \sqcup Y$ *is called a* ***complete bipartite graph*** *if for every* $x \in X$ *and* $y \in Y$, *we have* $xy \in E(G)$.

Note: • Such a graph is determined by the cardinalities of $n = |X|$ and $m = |Y|$, thus denoted $K_{n,m}$.

• A graph of the form $K_{1,m}$ is called a **star**.

Example (Complete bipartite graphs)**.**

Definition. • *For* $t \geqslant 2$, *a graph is* t-***partite*** *if* $V(G) = X_1 \sqcup \cdots \sqcup X_t$ *where* $X_1, \ldots, X_t$ *are non-empty independent subsets.*

• *A* t-***partite*** *graph is called a* ***complete t-partite graph*** *if every two vertices in different partite sets are joined by an edge.*

Note: Such complete t-partite graph is determined by the numbers $n_i = |X_i|$, thus denoted $K_{n_1,\ldots,n_t}$.

Example ((Complete) t-partite graphs)**.**

Definition. *The **complement** $\overline{G}$ of G is the graph with $V(\overline{G}) = V(G)$ and an edge $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.*

Example (Graphs and its complements)**.**

Theorem (1.11)**.** *If G is disconnected, then $\overline{G}$ is connected.*

Exercise. *Let G disconnected. If $u, v \in V(\overline{G})$, show that $d_{\overline{G}}(u, v) = 1$ or $d_{\overline{G}}(u, v) = 2$. What are the possible values for $\operatorname{diam}(\overline{G})$?*

Set-theoretical fact: If X and Y are two sets, one can always form their **disjoint union** $X \sqcup Y$ (a.k.a. coproduct of sets). This is the smallest possible set containing X and Y in a way such that $X \cap Y = \varnothing$.

Definition. *Fix graphs* G *and* H.

(a) *The **disjoint union*** $G \sqcup H$ *is the graph with* $V(G \sqcup H) = V(G) \sqcup V(H)$ *and* $E(G \sqcup H) = E(G) \sqcup E(H)$.

(b) *The **join*** $G \vee H$ *(or* $G + H$*) consists of* $G \sqcup H$ *and all edges joining a vertex of* G *and a vertex of* H. *In other words,*

$$G \vee H = (G \sqcup H) + \{uv \colon u \in V(G), v \in V(H)\}.$$

(c) *The **Cartesian product*** $G \,\square\, H$ *has:*

- *vertices* $V(G \,\square\, H) = V(G) \times V(H) = \{(u, v) \colon u \in V(G), v \in V(H)\}$;
- *an edge between* $(u, v) \neq (x, y)$ *if either* $\begin{cases} u = x \text{ and } vy \in E(H), \text{ or} \\ v = y \text{ and } ux \in E(G). \end{cases}$

Example (Union, join and Cartesian product)**.**

Definition. *(a) A **multigraph** is a graph where the edges can have multiplicities (we call them **multiedges**), and edges of the form* vv *are also allowed (these are called **loops**).*

*(b) A **digraph** is a pair* $D = (V, E)$ *where* V *is a nonempty set of vertices and* $E \subseteq \{(u, v) \in V \times V \colon u \neq v\}$. *The elements of* E*, called **arcs**, are ordered pairs of different elements of* V.

*(c) An **oriented digraph** is a digraph such that for each pair of vertices* u, v*, at most one of* (u, v) *and* (v, u) *is an arc.*

Example (Multigraphs, digraphs, oriented graphs)**.**

Definition. *For a vertex* v *in a graph* G*, the* ***degree*** *of* v *is* $\deg v =$ *number of edges incident with* v*.*

- We say that v is an {**even, odd**} **vertex** if $\deg v$ is {even, odd}.
- The **neighborhood** of v is the set $N(v) = \{u \in V(G) \colon uv \in E(G)\}$. Thus $\deg v = |N(v)|$.
- We say that v is an **isolated vertex** if $\deg v = 0$; we say that it is a **leaf** if $\deg v = 1$.
- The **minimum degree** of G is $\delta(G) = \min\{\deg v \colon v \in V(G)\}$; its **maximum degree** is $\Delta(G) = \max\{\deg v \colon v \in V(G)\}$.
- If G has n vertices, then $0 \leqslant \delta(G) \leqslant \deg v \leqslant \Delta(G) \leqslant n - 1$.

Example (Degrees, leafs, isolated vertices)**.**

Theorem ((2.1) The handshaking lemma)**.** *If* G *is a graph, then*

$$\sum_{v \in V(G)} \deg v = 2\,|E(G)|.$$

Example. *A graph has* 14 *vertices and* 27 *edges. The possible values for* $\deg v$ *are* 3, 4 *or* 5, *and exactly six vertices have degree* 4. *How many vertices have degree* 3 *and how many have degree* 5*?*

Corollary (2.3)**.** *Every graph has an even number of odd vertices.*