Pointed Hopf algebras over nonabelian groups

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Outline:

1) Introduction
2) The abelian case
3) The non-abelian case

Introduction: (Field $\Bbbk = \bar{\Bbbk}$, char $= 0$)

A Hopf Algebra: is an algebra (H, m, u) with additional structure

$$(H, m, u, \Delta, \varepsilon, S) \qquad \Delta: H \longrightarrow H \otimes H, \quad S: H \longrightarrow H, \quad \varepsilon: H \longrightarrow \Bbbk$$

In Mod H we have
- ⇝ Tensor products Δ
- ⇝ Unit object ε
- ⇝ Duals S

Examples: 1) group algebra $\Bbbk G$, $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ "Group-like"

2) Enveloping algebra $U(\mathfrak{g})$. $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ "Primitive"

3) Quantized enveloping algebras $U_q(\mathfrak{g})$, $u_q(\mathfrak{g})$

E.g: For $\mathfrak{g} = \mathfrak{sl}_2 = \langle f, h, e \rangle$, $U_q(\mathfrak{g}) = \langle F, K, E \,/\, KEK^{-1} = q^2 E,\ KFK^{-1} = q^{-2} F,\ [E,F] = \frac{K - K^{-1}}{q - q^{-1}} \rangle$

$$\Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F. \qquad \text{"Skew-Primitive"}$$

General problem: Classify finite (dimensional) Hopf algebras.

Out of reach, impose restrictions. Two main roads: ⇝ Semisimple (group Theory)
⇝ Pointed (Lie Theory)

Definition: A Hopf algebra H is pointed if every simple subcoalgebra is 1-dimensional.

i.e: The biggest cosemisimple part of H is (the group algebra of) the group

$$G(H) = \{ x \neq 0 \,/\, \Delta(x) = x \otimes x \}$$

Example: $U_q(\mathfrak{g})$, $u_q(\mathfrak{g})$ (The group is given by the maximal Torus).

Cartier - Konstant - Milnor - Moore:

If H is cocommutative then $H \simeq U(P) \# G$, P = primitives of H, G = group-likes of H

Radford, Majid:

If H is pointed then $\operatorname{gr} H \simeq R \# G$

Group-likes

$\rightsquigarrow$ Smash product (a.k.a. bosonization)

w.r.t coradical filtration
Group-likes $\subseteq$ Skew-primitives $\subseteq \dots$

Graded Hopf algebra in ${}^G_G\mathcal{YD}$,
a braided tensor category attached to G
Semisimple if G is finite

Andruskiewitsch - Schneider Philosophy:

If H is pointed and finite dimensional, R can be classified and then H can be recovered from $\operatorname{gr} H$.

Crucial tools $\rightsquigarrow$ To classify R: **Nichols algebras.**

$\rightsquigarrow$ To recover H from $\operatorname{gr} H$: Hopf **2-cocycles.**

Nichols algebras: Fix group G.

${}^G_G\mathcal{YD} = \{$ G-graded G-modules $V = \bigoplus_{g \in G} V_g$ / $h \cdot V_g \subseteq V_{hgh^{-1}}$ $\}$

This is a braided category: $c(v \otimes w) = g \cdot w \otimes v$ if $v \in V_g$.

Construction: Given $V \in {}^G_G\mathcal{YD}$, tensor algebra $T(V)$ becomes Hopf algebra in ${}^G_G\mathcal{YD}$ by $\Delta(v) = v \otimes 1 + 1 \otimes v$ $(v \in V)$. Define

$B(V)$ = Unique **quotient of $T(V)$** such that **V = primitives.** "Nichols algebra of V"

Crucial for the classification: If H is pointed and $\operatorname{gr} H \simeq R \# G(H)$, with $R = \bigoplus_{n \geq 0} R^n$, then

$$B(R^1) \hookrightarrow R$$

R^1 = "Infinitesimal Braiding of H"

Generation in degree 1 Conjecture [AS]

If H is finite dim, then $\mathcal{B}(R^1) = R$. Hence $\operatorname{gr} H \simeq \mathcal{B}(R^1) \# G$

Equivalently: H is generated by group-likes and skew-primitives.

To Recover H from $\operatorname{gr} H$:

A 2-cocycle σ on a Hopf algebra A is $\sigma : A \otimes A \to \Bbbk$ with some properties

$\rightsquigarrow$ Can build a new Hopf algebra A_σ (only mult. and antipode change using conjugation by σ)

A_σ is called a cocycle deformation of A

Masuoka philosophy: If H is such that $\operatorname{gr} H \simeq R \# G = \mathcal{B}(R^1) \# G$

Then $H \simeq (\mathcal{B}(R^1) \# G)_\sigma$ for some σ.

Andruskiewitsch - Schneider Program:

Fix a finite group G. To classify fin. dim. pointed Hopf algebras over G

1) Classify $V \in {}^G_G\mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$

2) For such V, give a presentation of $\mathcal{B}(V)$ (gens + rels).

3) Use 2) to show that the conjecture "$R = \mathcal{B}(R^1)$" holds.

4) If $\operatorname{gr} H \simeq \mathcal{B}(V) \# G$, show that $H \simeq (\mathcal{B}(V) \# G)_\sigma$ (Masuoka).

2) Case G abelian:

Simple objects of ${}^G_G\mathcal{YD} = \{ V = \bigoplus_{g\in G} V_g \ / \ h\cdot V_g \subseteq V_g \ \forall h,g \}$

are 1-dimensional, parametrized by pairs (g, χ) satisfying certain property.

$g \in G$, χ Character $G \longrightarrow \Bbbk$

$(g,\chi) \rightsquigarrow \Bbbk\{x\} \in {}^G_G\mathcal{YD}$ where $\deg x = g$, $h\cdot x = \chi(h)x$.

Now $\mathcal{B}(\Bbbk\{x\}) = \begin{cases} \Bbbk[x]/x^N & \text{if } \chi(g) \text{ root of unity order } N \\ \Bbbk[x] & \text{otherwise.} \end{cases}$

$\chi(g)$ determines the braiding: $c(x\otimes x) = \chi(g)\, x\otimes x$.

Really easy to decide which simples give fin. dim Nichols.

Next step: decide when $\bigoplus$ of simples give fin. dim. Nichols.

Heckenberger ($\sim$08): Introduced root systems for these non-simple Nichols algebras, reversing Lusztig's construction of the Weyl group action in the quantum group.

From now on, given $V \in {}^G_G\mathcal{YD}$ we only care about the braiding $c: V\otimes V \longrightarrow V\otimes V$ ($\mathcal{B}(V)$ as a bialgebra only depends on c)

Construction Given $q = (q_{ij})_1^n$, $q_{ij} \in \Bbbk^\times$ we build

$V = \Bbbk\{x_1, \dots, x_n\}$ with $c(x_i\otimes x_j) = q_{ij}\, x_j \otimes x_i$ "Diagonal braiding"

For this q one defines a "Dynkin diagram"

Graph with labeled vertices $\overset{q_{11}}{\underset{1}{\bullet}}$, $\overset{q_{22}}{\underset{2}{\bullet}}$, ..., $\overset{q_{nn}}{\underset{n}{\bullet}}$.

edge between $i \neq j$ iff $q_{ij}q_{ji} \neq 1$, with that label

$$\cdots \overset{q_{ii}}{\underset{i}{\bullet}} \xrightarrow{\quad q_{ij}q_{ji} \quad} \overset{q_{jj}}{\underset{j}{\bullet}} \cdots$$

If G is abelian, all $V \in {}^G_G\mathcal{YD}$ are of this form!

(The possibilities for $q = (q_{ij})$ depend on $\hat{G}$)

General fact (Lyndon words Theory)

$B(V)$ admits a PBW-Type basis of the form

$$\left\{ \prod_{\alpha \in \Delta(V)} x_\alpha^{n_\alpha} \ : \ 0 \leq n_\alpha < N_\alpha \right\}$$

Where each x_α is $\mathbb{Z}^n$-homogeneous of degree α

"$\Delta(V)$ = The set of roots of V."

Major difference with Lusztig's construction:

A matrix $q = (q_{ij})$ could be reflected to a different matrix $P = (P_{ij})$, and

now Lusztig's isomorphism $T_i : B(q) \# \mathbb{Z}^n \xrightarrow{\sim} B(P) \# \mathbb{Z}^n$ (as algebras)

So now we have a Weyl groupoid

And one obtains a "root system" considering all the sets

$\Delta(P)$ where $P = (P_{ij})$ runs over all reflections of q.

The same phenomenom was known for certain Lie superalgebras (Serganova's odd roots)

Example: Let $(d_i a_{ij})$ symmetrized finite Cartan matrix. with Lie alg. $\mathfrak{g}$

Let q root of unity (of odd order) and define $q = \left(q^{d_i a_{ij}}\right)_{i,j}$

Then $B(q) \simeq u_q^+(\mathfrak{g})$, Positive part of small quantum group.

Example: Let q of odd order and q with Dynkin diagram

Then $B(q) \simeq u_q^+ \mathfrak{sl}(m|n)$

Heckenberger '08 : classified all q with $\dim \mathcal{B}(q) < \infty$.

Angiono '13 Generators and relations for such $\mathcal{B}(q)$.

Used them to prove "$R = \mathcal{B}(R^1)$"

Angiono – García Iglesias '19 : if $\mathrm{gr}\, H \simeq \mathcal{B}(V)\#G \Rightarrow H \simeq (\mathcal{B}(V)\#G)_\sigma$

+ recipe to compute $(\mathcal{B}(V)\#G)_\sigma$.

(Andruskiewitsch–Schneider '09 proved all of these assuming $|G|$ is not divisible by 2, 3, 5, 7)

Upshot: finite dimensional pointed Hopf algebras over abelian groups are classified.

3) Case G nonabelian (Joint with Angiono, Lentner)

Simples of ${}^G_G\mathcal{YD}$: parametrized by pairs $(\underbrace{g^G}_{\text{Conjugacy class of } g\in G}, \underbrace{\rho}_{\text{Irrep of } G^g\text{, The centralizer of } g})$

Such a pair induces $M(g^G, \rho) = \mathrm{Ind}_{G^g}^G(\rho) \in \mathrm{Irr}({}^G_G\mathcal{YD})$

Major obstruction: Very hard to classify simples in ${}^G_G\mathcal{YD}$ with finite dimensional Nichols algebra.

Example: $G = S_n$, $(12)^{S_n}$ = Conjugacy class of transpositions.

ρ = Irrep of $S_n^{(12)}$ built from sign.

Then $\dim \mathcal{B}((12), \rho) = \begin{cases} 12 & n=3 \\ 576 & n=4 \\ 8294400 & n=5 \\ \text{conjecturally } \infty, & n>5 \end{cases}$

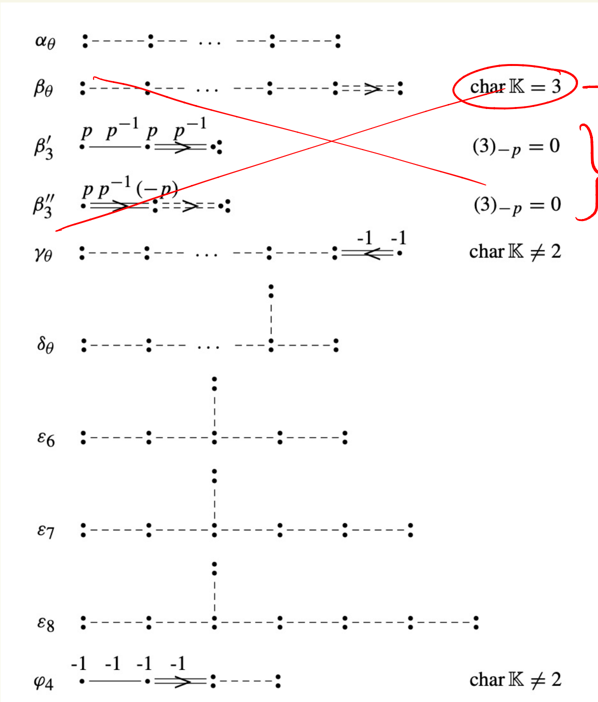
"Fomin–Kirillov algebras"

Heckenberger-Vendramin '18 :

- Classified nonsimples $\bigoplus_i V_i \in {}^G_G\mathcal{YD}$ such that $\dim \mathcal{B}(\bigoplus_i V_i) < \infty$.
- Described the possibilities for G.

Crucial Tool: Weyl groupoids and root systems can be constructed

Picture of the classification:



→ Not considered here

Exceptions coming from previous 12-dimensional Nichols algebra over S_3

- # of points = dim of simple
- Continuos lines $\underset{V_1}{:} = \underset{V_2}{.}$ mean that Support of V_1 (G-degrees happening on V_1) commutes with support of V_2
- Number of lines: defined using vanishing exponents for the braided adjoint action ⇝ Can define a "Cartan matrix"
- Arrows: defined as for simple Lie algebras using the Cartan matrix above.

Our goal: Classify pointed Hopf algebras with Nichols algebra of type $\alpha, \gamma, \delta, \varepsilon$, or φ (in char 0)

Roadmap: Consider $V = \bigoplus_i V_i$ with Nichols algebra as above.

1) G fits in a central extension of $\mathbb{Z}_2$ by an abelian group Γ.

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow G \longrightarrow \Gamma \longrightarrow 1$$

2) Folding construction (Lentner) If $\mathbb{Z}_2$ acts trivially on V, the Nichols algebra is built using Nichols algebras over abelian groups and Dynkin diagram automorphisms.

3) If the rank (# simple factors) is ≥ 4, one can show that $\mathbb{Z}_2$ acts trivially.

4) In general one can "trivialize" the action of $\mathbb{Z}_2$ on V using a certain endofunctor $F_\eta : {}^G_G\mathcal{YD} \longrightarrow {}^G_G\mathcal{YD}$ constructed from a 2-cocycle $\eta \in Z^2(G, \Bbbk^\times)$.

We show that there exists η such that $\mathbb{Z}_2$ acts trivially on $F_\eta(V)$.

5) Use what is known for $\mathcal{B}(F_\eta(V))$ (here the group is abelian) to better understand $\mathcal{B}(V)$

Summary of results:

Let H finite-dimensional Hopf algebra over a non-abelian group G and with Nichols algebra $\mathcal{B}(V)$ of type $\alpha, \gamma, \delta, \varepsilon$ or φ, so that $\operatorname{gr} H \simeq R \# G$ where $\mathcal{B}(V) \hookrightarrow R$. Then

1) $R = \mathcal{B}(V)$ (ie: H is generated by group-likes and skew-primitives)

2) Defining relations for $\mathcal{B}(V)$ are known.

3) There exists a Hopf 2-cocycle σ for $\mathcal{B}(V)\#G$ such that

$$H \simeq (\mathcal{B}(V)\#G)_\sigma$$

Thank you!