22. Wednesday, August 7th. Compacteness. Compact subspaces of $\mathbb{R}$.

Last Time:

Theorem 2: A product $X \times Y$ is compact if and only if X and Y are compact

To prove it, we will need the following:

The Tube Lemma: Consider $X \times Y$ where Y is compact. Fix $x_0 \in X$

If $N \subseteq X \times Y$ is an open set containing $\{x_0\} \times Y$, Then There exists

an open set U of X containing x_0 such That $U \times Y \subseteq N$

Proof of Thm 2:

$(\Rightarrow)$ If $X \times Y$ is compact, since The projections

$P_X : X \times Y \longrightarrow X$ and $P_Y : X \times Y \longrightarrow Y$ are continuous and surjective,

we deduce That X and Y are compact.

($\Leftarrow$) Assume X and Y are compact. We want to show that $X \times Y$ is compact, so take an open cover $\{W_i\}_{i \in \Lambda}$ of $X \times Y$.

For each $x \in X$, we have $\{x\} \times Y \subseteq \bigcup_{i \in \Lambda} W_i$. Since $\{x\} \times Y$ is homeomorphic to Y, which is compact, we know that $\{x\} \times Y$ is compact.

Thus there exists a finite subset $I(x) \subseteq \Lambda$ such that $\{x\} \times Y \subseteq \bigcup_{i \in I(x)} W_i = N(x)$ → Notation

Since Y is compact, we can apply the Tube Lemma and deduce that there exists some open $U(x)$ in X with $\{x\} \times Y \subseteq U(x) \times Y \subseteq N(x)$

This holds for every $x \in X$. Thus we have an open cover $\{U(x)\}_{x \in X}$ of X

Since X is compact, there exist a finite subset $A \subseteq X$ such that $X = \bigcup_{x \in A} U(x)$ Therefore

$$X \times Y = \left(\bigcup_{x \in A} U(x)\right) \times Y = \bigcup_{x \in A} U(x) \times Y \subseteq \bigcup_{x \in A} N(x) = \underbrace{\bigcup_{x \in A} \bigcup_{i \in I(x)}}_{\text{finite!}} W_i$$

So any open cover of $X \times Y$ admits a finite subcover $\square$

<u>Note</u>: Thm still holds for arbitrary products with the product topology (Tychonoff's Thm Chapter 5)

Compactness in $\mathbb{R}$

Recall: Proved (a,b), $[a,b)$, $(a,b]$, (a,∞), $[a,\infty)$, $(-\infty,a)$, $(-\infty,a]$ are not compact.

Theorem 1: For any $a<b$, The interval $[a,b] \subseteq \mathbb{R}$ is compact.

Proof: Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ open sets in $\mathbb{R}$ with $[a,b] \subseteq \bigcup_{i\in\Lambda} U_i$. WTS: There exists a finite subcover of $\mathcal{U}$.

Consider The set

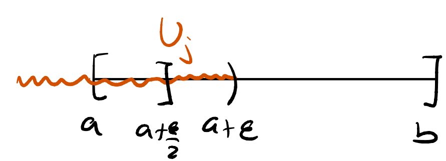
$$C = \{ x \in (a,b] \mid [a,x] \text{ admits a finite subcover of } \mathcal{U}. \}$$

Claim 1: C is non-empty and has an upper bound, Thus it has a supremum $s = \sup C$.

Proof: Since $a \in [a,b] \subseteq \bigcup_{i\in\Lambda} U_i$, we have $a \in U_j$ for some $j \in \Lambda$.

Since U_j is open in $\mathbb{R}$, $\exists$ $\varepsilon > 0$ such That $(a-\varepsilon, a+\varepsilon) \subseteq U_j$. In particular

$[a, a+\frac{\varepsilon}{2}] \subseteq U_j$, Thus $a + \frac{\varepsilon}{2} \in C$, which is non empty.



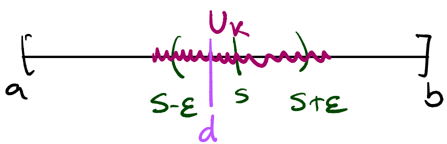
Also, b is an upper bound for C, because $C \subseteq (a,b]$.

$\square$ Claim 1

Claim 2: $s = \sup C \in C$.

Proof: First, we show $s \in (a, b]$. The inequality $a < c$ follows using that $a + \frac{\varepsilon}{2} \in C$ for $\varepsilon > 0$ from previous claim. The inequality $s \leq b$ follows because b is an upper bound for C and s is the smallest one.

Second we show $[a, s]$ admits a finite subcover of $\mathcal{U}$. Since $s \in (a, b] \subseteq \bigcup_{i \in \Lambda} U_i$, we have $s \in U_k$ for some $k \in \Lambda$. Since $U_k \subseteq \mathbb{R}$ is open, there exists $\varepsilon > 0$ such that $(s - \varepsilon, s + \varepsilon) \subseteq U_k$, and since $a < s$, we can assume that $a < s - \varepsilon$.



Now, since $s - \varepsilon < s = \sup C$, we know $s - \varepsilon$ is <u>not</u> an upper bound for C. Hence $\exists\, d \in C \cap (s - \varepsilon, s]$. But then $[a, d]$ admits a finite subcover (because $d \in C$). Since $[d, s] \subseteq U_k$, also $[a, s] = [a, d] \cup [d, s]$ admits a finite subcover.

$\square_{\text{Claim 2}}$

Claim 3: $s = b$

Proof: We proved $s \leq b$. Assuming for a contradiction that $s < b$, on claim 2 we could have assumed also $s + \varepsilon < b$, so $[s, s + \frac{\varepsilon}{2}] \subseteq U_k$. Therefore

$$[a, s + \tfrac{\varepsilon}{2}] = [a, s] \cup [s, s + \tfrac{\varepsilon}{2}]$$

also admits a finite subcover. But this means that $s + \frac{\varepsilon}{2} \in C$, contradicting that s is an upper bound for C. $\square$ Claim 3

Therefore, $[a, s] = [a, b]$ admits a finite subcover for $\mathcal{U}$.

This works for any open cover, hence $[a, b]$ is compact $\square$.