Instructions for the Midterm

- You may bring one standard 8.5x11 sheet of paper with handwritten notes.
- All materials through Feb 23 might be evaluated. You will not be tested on vertex/edge connectivity or Menger's theorem. You will not be tested on topics involving linear algebra (adjacency matrix, the Matrix Tree Theorem).
- The midterm will consist of **four questions selected from the list below**.
- Each question will be worth 25 pts.

Sections 1.1-1.4

Problem 1 (25pts)**.** Give an example of a connected graph G containing three vertices u, v, and w such that $d(u,v) = d(u,w) = d(v,w) = \mathrm{diam}(G) = 3$.

Problem 2 (25pts)**.** Let $P = (u = v_0, v_1, \ldots, v_k = v)$, $k \geqslant 1$, be a u–v geodesic in a connected graph G. Prove that $d(u, v_i) = i$ for each integer i with $1 \leqslant i \leqslant k$.

Problem 3 (25pts)**.** Prove that if P and Q are two longest paths in a connected graph, then P and Q have at least one vertex in common.

Problem 4 (25pts)**.** Let G be a disconnected graph. Prove that the diameter of $\overline{G}$ is at most 2.

Problem 5 (25pts)**.** Let G be a graph of order $n \geqslant 5$. Prove that at most one of G and $\overline{G}$ is bipartite.

Sections 2.1-2.4

Problem 6 (25pts)**.** Prove that for every graph G, there are vertices u and v such that $\deg u = \deg v$.

Problem 7 (25pts)**.** Prove that if a graph of order $3n$ ($n \geqslant 1$) has n vertices of each of the degrees $n-1$, n, and $n+1$, then n is even.

Problem 8 (25pts)**.** Show that if G is a disconnected graph containing exactly two odd-degree vertices, then these odd-degree vertices must be in the same component of G.

Problem 9 (25pts)**.** Prove that if G is a graph of order n such that $\Delta(G) + \delta(G) \geqslant n - 1$, then G is connected and $\mathrm{diam}(G) \leqslant 4$. Show that the bound $n - 1$ is sharp.

Problem 10 (25pts)**.** A certain connected graph G has the property that for every two vertices u and v of G, either

- the length of each u–v path is even, or
- the length of each u–v path is odd.

Prove that G is bipartite.

Problem 11 (20 + 5 pts)**.** Let G be a connected graph and consider a function $f\colon V(G) \to X$ where X is some arbitrary set. Prove that if f is *not* a constant function, then there is an edge $uv \in E(G)$ such that $f(u) \neq f(v)$.

Deduce that if G is not regular, then G contains adjacent vertices u and v such that $\deg u \neq \deg v$.

Problem 12 (12 + 13 pts)**.** (a) Let v be a vertex of a graph G. Show that if $G - v$ is 3-regular, then G has odd order.

(b) Let G be an r-regular graph, where r is odd. Show that G does not contain any component of odd order.

Problem 13 (12 + 13 pts). (a) Show that a graph G is regular if and only if $\overline{G}$ is regular.

(b) Show that if G and $\overline{G}$ are both r-regular for some nonnegative integer r, then G has odd order.

Problem 14 (25pts). Let G be a graph on n vertices. Prove that if $\Delta(G) + \delta(G) \geqslant n - 1$ then G is connected.

Problem 15 (25pts). Let G be a graph on n vertices. Prove the following:

(a) $\delta(G) + \delta(\overline{G}) \leqslant n - 1$,

(b) $\delta(G) + \delta(\overline{G}) = n - 1$ if and only if G is regular.

Problem 16 (8 + 8 + 9 pts). Determine which of the following sequences are graphical. For each that are graphical, construct a graph, as in Example 2.11, for which the given sequence is a degree sequence of the graph.

(a) $s_1 : 5, 3, 3, 3, 3, 2, 2, 2, 1$ (b) $s_2 : 6, 3, 3, 3, 3, 2, 2, 2, 2, 1, 1$ (c) $s_3 : 7, 6, 5, 4, 4, 3, 2, 1$

Problem 17 (25pts). Prove that for every integer x with $0 \leqslant x \leqslant 5$, the sequence $x, 1, 2, 3, 5, 5$ is not graphical.

Problem 18 (25pts). For which integers x ($0 \leqslant x \leqslant 7$), if any, is the sequence $7, 6, 5, 4, 3, 2, 1, x$ graphical?

Problem 19 (25pts). If the sequence $x, 7, 7, 5, 5, 4, 3, 2$ is graphical, then what are the possible values of x ($0 \leqslant x \leqslant 7$)?

Sections 3.1-3.2

Problem 20 (25pts). Does there exist a disconnected self-complementary graph?

Problem 21 (25pts). A graph G is called *self-complementary* if $G \cong \overline{G}$. Prove that if G is a self-complementary graph on n vertices, then n is congruent to either 0 or 1 modulo 4. Exhibit an example of a self-complementary graph on 4 vertices and an example of a self-complementary graph on 5 vertices.

Problem 22 (25pts). Let G be a self-complementary graph of order $n = 4k$, where $k \geqslant 1$. Let $U = \{v : \deg v \leqslant n/2\}$ and $W = \{v : \deg v \geqslant n/2\}$. Prove that if $|U| = |W|$, then G contains no vertex v such that $\deg v = n/2$.

1 Sections 4.1-4.3

Problem 23 (25pts). Prove that every connected graph all of whose vertices have even degrees contains no bridges.

Problem 24 (25pts). Prove that if uv is a bridge in a graph G, then there is a unique u–v path in G.

Problem 25 (25pts). Let G be a connected graph and let e_1 and e_2 be two edges of G. Prove that $G - e_1 - e_2$ has three components if and only if both e_1 and e_2 are bridges in G.

Problem 26 (25pts). Prove that if G is a graph with $\delta(G) \geqslant 2$, then G contains a cycle.

Problem 27 (25pts)**.** A certain tree T of order n contains only vertices of degree 1 and 3. Show that T contains $(n-2)/2$ vertices of degree 3.

Problem 28 (20 + 5 pts)**.** Let T be a tree on n vertices with m edges. Let n_i denote the number vertices of degree i for $i \geqslant 1$. Thus $n = \sum_i n_i$ and $2(n-1) = 2m = \sum_i n_i$.

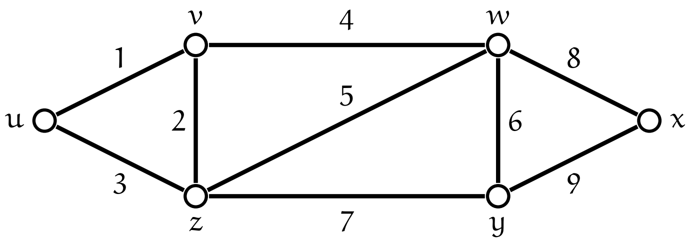
(a) Prove that $n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots$

(b) A tree T has three vertices of degree 2, five vertices of degree 3, two vertices of degree 4 and no vertices of degree 5 or more. How many leaves does T have?

Problem 29 (25pts)**.** Find all trees T such that $\overline{T}$ is also a tree.

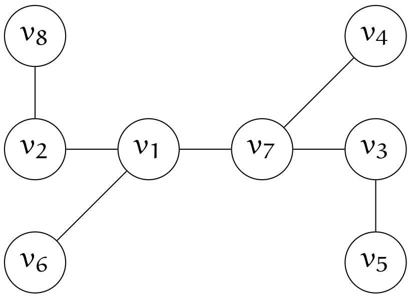
Problem 30 (25pts)**.** Prove that an edge e of a connected graph is a bridge if and only if e belongs to every spanning tree of G.

Problem 31 (25pts)**.** Apply both Kruskal's and Prim's Algorithms to find a minimum spanning tree in the following graph. In each case, show how this tree is constructed.

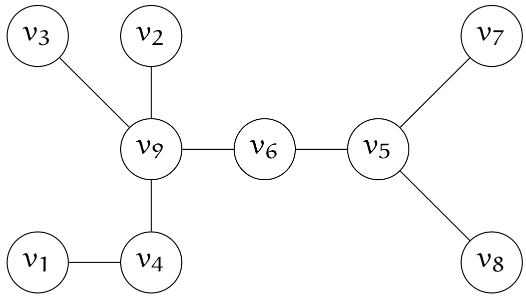


Problem 32 (12 + 13 pts)**.** Determine the Prüfer code of the following trees, using the ordering $v_1 \leqslant v_2 \leqslant \ldots$ for the vertices.

(a)



(b)



Problem 33 (12 + 13 pts)**.** For the following sequences, draw a picture of the (labeled) tree which has that sequence as its Prüfer code (using the standard ordering of the integers). Your tree should have vertex-set $\{v_1, \ldots, v_n\}$ for some integer n.

(a) $(v_3, v_7, v_3, v_1, v_5, v_3, v_3)$

(b) $(v_2, v_2, v_1, v_4, v_1, v_4, v_3)$

2 Sections 5.1-5.2

Problem 34 (25pts)**.** Prove that if v is a cut-vertex of a graph G, then v is not a cut-vertex of the complement $\overline{G}$ of G.

Problem 35 (25pts)**.** Prove that a 3-regular graph G has a cut-vertex if and only if G has a bridge.

Problem 36 (25pts)**.** Prove that if T is a tree of order at least 3, then T contains a cut-vertex v such that every vertex adjacent to v, with at most one exception, is a leaf.