Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (0.5 + 0.5 pts)**.** Recall that K_{n_1,n_2,n_3} is the complete tripartite graph with parts of sizes n_1, n_2, n_3.

Fix any positive integer n.

(a) Prove that $K_{n,2n,3n}$ is Hamiltonian.

(b) Prove that $K_{n,2n,3n+1}$ is not Hamiltonian.

Problem 2 (1 + 2 pts)**.** Fix any integer $n \geqslant 3$.

(a) Construct an n-vertex graph G with $\binom{n-1}{2} + 1$ many edges such that G is *not* Hamiltonian.

(Note: You must construct such a graph for *every* $n \geqslant 3$.)

(b) Prove that if G is an n-vertex graph with at least $\binom{n-1}{2} + 2$ many edges, then G is Hamiltonian.

(Hint: Ore. Also, a graph on k-vertices has at most $\binom{k}{2}$ edges, and $\binom{n-1}{2} - \binom{n-2}{2} = n - 2$)

Problem 3 (2pts)**.** Let G be a graph on $n \geqslant 4$ vertices with the property that $N(u) \cup N(v) \supseteq V(G) \setminus \{u, v\}$ for every $u \neq v \in V(G)$. Prove that G is Hamiltonian.

Problem 4 (2pts)**.** Prove that $\alpha(G) + \beta(G) = n$ for any n-vertex graph G.

(Note: You don't need to know anything about matchings to prove this.)

Problem 5 (2 pts)**.** Let G be a bipartite graph on n vertices. Show that $\alpha(G) = n/2$ if and only if G has a perfect matching.