Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1 pt)**.** Recall that a *directed graph* (or digraph) is a pair $D = (V, E)$ where V is a set and $E \subseteq \{(u, v) \in V \times V \colon u \neq v\}$. For $u, v \in V$, a u-v diwalk in D is a sequence $(u = v_0, \ldots, v_k = v)$ such that $(v_i, v_{i+1}) \in E$ for all $i \in \{0, \ldots, k-1\}$.

Consider the relation R on V where $u \mathrel{R} v$ iff there is a u-v diwalk. Give an example of a digraph D where R is **not** an equivalence relation. Justify your answer. (Feel free to draw a picture of D)

Problem 2 (2 pts)**.** Let G be a connected graph and consider a function $f \colon V(G) \to X$ where X is some arbitrary set. Prove that if f is *not* a constant function, then there is an edge $uv \in E(G)$ such that $f(u) \neq f(v)$.

Problem 3 (2 pts)**.** Prove that every graph on at least two vertices has a pair of vertices with the same degree.

Problem 4 (2 pts)**.** Prove that if G is a graph with $\delta(G) \geqslant 2$, then G must contain a cycle.

Problem 5 (3 pts)**.** Let G be a graph and let A be an independent set of G. Prove that

$$\sum_{v \in A} \deg v \leqslant |E(G)|$$

with equality if and only if G is bipartite with parts A and $V(G) \setminus A$.
(Especially in this problem, be sure to carefully justify all steps in your argument)
Hint: consider the sets $E_v = \{\text{edges incident with } v\}$ and $\widehat{E} = \bigcup_{v \in A} E_v$. For the equality, decide when is $\widehat{E} = E(G)$.