Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2 pts)**.** Let S be a finite set of integers and let G be the graph on vertex set S where for any $x, y \in S$, $xy \in E(G)$ if and only if $x + y$ is odd.

1. Prove that G is a bipartite graph for any such S.

2. If $S = \{1, 2, \ldots, 100\}$, what is $|E(G)|$? Justify your answer.

Problem 2 (2 pts)**.** Suppose that $(u = v_0, v_1, \ldots, v_k = v)$ is a u–v geodesic. Prove that $d(u, v_i) = i$ for all $i \in \{0, \ldots, k\}$.

Problem 3 (2 pts)**.** Let G be a graph. For two non-empty subsets $A, B \subseteq V(G)$, an A–B path is a path in G which connects some vertex of A to some vertex of B. Prove that if P is a minim**al** A–B path, then P contains exactly one vertex from A and contains exactly one vertex from B.

Problem 4 (2 pts)**.** Let G be a connected graph. Prove that any two maxim**um** paths in G must share some vertex.

Problem 5 (2 pts)**.** Prove that a graph G is bipartite if and only if every subgraph H of G has an independent set consisting of at least half of $V(H)$. (recall that an independent set is a set of vertices which induce no edges)