3. Monday, June 24. Bases and subbases (§13)

Last Time: A Topology on X is $\mathcal{T} \subseteq \mathcal{P}(X)$ st:
1) $\emptyset, X \in \mathcal{T}$
2) Arbitrary $\cup$
3) Finite $\cap$

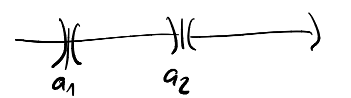
Definition 1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a set X.

If $\mathcal{T}_1 \subseteq \mathcal{T}_2$ we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is finer than $\mathcal{T}_2$

We say $\mathcal{T}_1$ and $\mathcal{T}_2$ are comparable if either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Example: On a set X, $\mathcal{T}_{Triv}$ is coarser than any other topology and $\mathcal{T}_{dis}$ is finer than any other

Exercise: Are $\mathbb{R}_{std}$ and $\mathbb{R}_{cf}$ comparable?

In $\mathbb{R}_{cf}$, opens look like $U = \mathbb{R} \setminus \{a_1, \dots, a_n\}$ 

Bases and subbases

It can be hard to specify all elements in a topology and verify they satisfy the axioms.
Want an easier way to describe and build topologies.
Both can be achieved using bases (Replace $B(x,\epsilon)$'s in $\mathbb{R}_{std}$)

Lemma 2: Let X be a set and let $\mathcal{B} \subseteq P(X)$. Then we have an equality

$$\underbrace{\left\{ U \subseteq X \mid U = \bigcup_{i\in\Lambda} B_i \text{ for some } \{B_i\} \subseteq \mathcal{B} \right\}}_{\text{"Unions of sets in } \mathcal{B}\text{"}} = \left\{ U \subseteq X \mid \forall x \in U \ \exists B \in \mathcal{B} \text{ st } x \in B \subseteq U \right\},$$

We denote this common set by $T(\mathcal{B})$

Proof: $[\subseteq]$ Assume $U = \bigcup_{i\in\Lambda} B_i$ where $B_i \in \mathcal{B}$ $\forall i \in \Lambda$. Let $x \in U$.
Since $U = \bigcup_{i\in\Lambda} B_i$, we have $x \in B_i$ for some $i \in \Lambda$. Hence $x \in B_i \subseteq U$ ✓

$[\supseteq]$ Let U in RHS of equality. For each $x \in U$, fix some $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$
We claim that $U = \bigcup_{x\in U} B_x$. For $[\subseteq]$, given $y \in U$ we have $y \in B_y \subseteq \bigcup_{x\in U} B_x$
For $[\supseteq]$, just note that $B_x \subseteq U$ $\forall x \in U$, so also $\bigcup_{x\in U} B_x \subseteq U$. $\square$

Definition 3: Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{B} \subseteq \mathcal{P}(X)$.
We say that $\mathcal{B}$ is a basis for $\mathcal{T}$ if $\mathcal{T}(\mathcal{B}) = \mathcal{T}$.
i.e.: open sets are unions of elements in $\mathcal{B}$.
We also say that $\mathcal{B}$ generates $\mathcal{T}$.

Example: Discrete topology on a set X, has basis $\mathcal{B} = \{ \{x\} \mid x \in X \}$

Example: Trivial topology on a set X has basis $\mathcal{B} = \{ X \}$

Example: $\mathbb{R}$ with standard topology $\mathcal{T}_{std}$.
A basis for this topology is $\mathcal{B} = \{ B(x, \varepsilon) \mid x \in \mathbb{R}, \varepsilon > 0 \}$
This is by definition:

$$\mathcal{T}_{std} = \{ U \subseteq \mathbb{R} \mid \forall x \in U \ \exists \varepsilon > 0 \text{ st } B(x, \varepsilon) \subseteq U \} \overset{\text{Lemma 2.}}{=} \mathcal{T}(\mathcal{B})$$

Notice also that any open interval (a,b) is of the form $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for some $x \in \mathbb{R}$ and some $\varepsilon > 0$, just take $x = \frac{a+b}{2}$ and $\varepsilon = \frac{b-a}{2}$.

So $\mathcal{B} = \{ (a,b) \subseteq \mathbb{R} \mid a < b \}$ is a basis for $\mathbb{R}_{std}$.

So far, used bases to describe topologies. How do we define topologies?

Q: Let X be a set and $\mathcal{B} \subseteq \mathcal{P}(X)$. When is $\mathcal{T}(\mathcal{B})$ a topology on X?

Proposition 4: Let X be a set and let $\mathcal{B} \subseteq \mathcal{P}(X)$.

Then $\mathcal{T}(\mathcal{B})$ is a topology on X if and only if:

1) $X = \bigcup_{B \in \mathcal{B}} B$, and

2) $\forall\, B_1, B_2 \in \mathcal{B}$ and $\forall\, x \in B_1 \cap B_2$, $\exists\, B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$

Proof: [$\Rightarrow$] Assume $\mathcal{T}(\mathcal{B})$ is a topology. Then, by axiom 1, $X \in \mathcal{T}(\mathcal{B})$, so X is the union of some elements in $\mathcal{B}$, hence also the union of all of them.

To prove (2), let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. Since $B_1 \cap B_2$ is open, we have $B_1 \cap B_2 \in \mathcal{T}(\mathcal{B})$. Hence, by the second description of $\mathcal{T}(\mathcal{B})$ (from lemma 2), we know $\exists\, B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$, as desired. TBC.