Problem 1. Let X be a topological space and let $Y \subseteq X$ endowed with the induced topology. Prove that Y is connected if and only if for every open sets U, V of X such that $Y \subseteq U \cup V$ and $Y \cap U \cap V = \varnothing$, we have either $Y \subseteq U$ or $Y \subseteq V$.

Problem 2. Determine when a set X endowed with the cofinite topology is connected. Prove the resulting statement.

Problem 3. Prove the following:

1. A product of path-connected spaces is connected.
2. If $f \colon X \to Y$ is continuous and X is path-connected, then $f(X)$ is path-connected.
3. Any quotient of a path-connected space is path-connected.

Problem 4. A subset $X \subseteq \mathbb{R}^n$ is called *convex* if for every $x, y \in X$ and every $t \in [0, 1]$, we have $tx + (1-t)y \in X$ (in other words, the straight line segment from x to y is contained in X). Show that convex sets are path-connected.

Problem 5. Prove the following:

1. For any $n > 1$, the space $\mathbb{R}^n \backslash \{0\}$ is path connected.
2. For any $r \in \mathbb{R}$, the space $\mathbb{R} \backslash \{r\}$ is not connected.
3. For any $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.

Problem 6. For $k \geqslant 1$, let S^k denote the unit sphere in $\mathbb{R}^{k+1}$, namely

$$S^k = \{(x_1, \dots, x_{k+1}) \in \mathbb{R}^{k+1} \mid ||x|| = x_1^2 + \cdots + x_{k+1}^2 = 1\}.$$

For every $n \geqslant 2$, prove that $\mathbb{R}^n \backslash \{0\}$ is homeomorphic to $S^{n-1} \times \mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ denotes the space of positive real numbers. You should write a formula for a homeomorphism and it's inverse, but you need not verify continuity.

Deduce that the sphere S^n is path-connected for any $n \geqslant 1$.

Problem 7. Let X be a topological space.

1. Define a relation on X by $x \sim y$ if and only if there exists a connected subspace $Y \subseteq X$ that contains both x and y, for every $x, y \in X$. Verify that this is an equivalence relation. The equivalence classes are called the *connected components* of X.
2. Define a relation on X by $x \sim y$ if and only if there exists a path in X from x to y, for every $x, y \in X$. Verify that this is an equivalence relation. The equivalence classes are called the *path components* of X.