SMALL QUANTUM GROUPS OF TYPE SUPER A AND KNOTS INVARIANTS

GUILLERMO SANMARCO (Iowa State University)
Joint with ROBERT LAUGWITZ (University of Nottingham)

Motivation / History

(Field $k = \bar{k}$, char 0)

Hopf algebras ⇝ Modular fusion categories

- **Fusion** (finite, tensor, s.simple)
- \+ **Braided** ($c_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$)
- \+ **Ribbon** (Twist $\theta_x : X \to X$, $\theta_x \otimes \theta_y \circ c^2 = \theta_{x \otimes y}$, $(\theta_x)^* = \theta_{x^*}$)
- \+ **Non-degeneracy** of S-matrix

Modular fusion categories ⇝ Applications

Applications: Low-dim top., top. phases of matter, quantum computing...

Hopf algebras ⇝ Examples

Examples:

- Semisimplification of Tilting modules of $U_q(g)$
- Via Reps
- Via Reps of doubles

Drop semisimplicity:

Definition (Lyubashenko)

Modular Tensor category (MTC) = Finite Tensor + Braided ($c_{X,Y}: X\otimes Y \xrightarrow{\sim} Y\otimes X$) + Ribbon (Twist $\theta_X : X \to X$, $\theta_X \otimes \theta_Y \circ c^2 = \theta_{X\otimes Y}$, $(\theta_X)^* = \theta_{X^*}$) + Non-degenerate (Trivial Müger center)

Shimizu: Many equivalent characterizations

Applications:

Low-dim Topology, some field Theories, mapping class groups...

Key source for MTCs: Drinfeld centers

Upshot: for centers, main obstruction for modularity is ribbonality

Ribbonality of doubles

[Kauffman-Radford] For H fin. dim Hopf,

Ribbon elements of $D(H)$ $\longleftrightarrow$ $\left\{ (\beta, k) \,/\, \beta^2 = \alpha_H, \; k^2 = g_H, \; S^2(h) = \dots \right\}$

Distinguished group-likes of H

$p\lambda = p(g_H)\lambda$, $p \in H^*$,

Right integral of H^*

$\alpha_H(h)\Lambda = \Lambda h$, $h \in H$

Left integral of H

Ribbonality of Centers

[Shimizu] For $\mathcal{C}$ finite tensor

Distinguished invertible of $\mathcal{C}$

Radford iso $D \otimes X \xrightarrow{\sim} X^{4*} \otimes D$

1) Ribbon structures on $Z(\mathcal{C})$ $\longleftrightarrow$ $\{$ certain "square roots" of (D, ζ_D) $\}$ $=: \mathrm{SqrT}_{\mathcal{C}}(D, \zeta_D)$

2) If $\mathcal{C}$ is spherical $\Longrightarrow \mathrm{SqrT}_{\mathcal{C}}(D, \zeta_D)$ not empty

$\Longrightarrow Z(\mathcal{C})$ modular

Generalizes Müger's result in the fusion setting

Warning: [Douglas–Schommer-Pries–Snyder]

- $\mathcal{C}$ is pivotal if $\exists\ j : X \xrightarrow{\sim} X^{**}$
- $(\mathcal{C}, j)$ is Trace-spherical if $\dim_j X = \dim_j X^*$
- $(\mathcal{C}, j)$ is spherical if $\exists$ iso $\nu : \mathbb{1} \xrightarrow{\sim} D$

s.t.

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "\nu"] & X^{4*} \arrow[d, "\nu"] \\
D \otimes X \arrow[r, "\zeta_D"] & X^{*4} \otimes D
\end{tikzcd}$$

Ribbonality of relative centers $Z_{\mathcal{A}}(\mathcal{C})$

$c_{X,Y} : X\otimes Y \to Y\otimes X$

$c^{op}_{X,Y} : c^{-1}_{Y,X} : X\otimes Y \xrightarrow{bop} Y\otimes X$

$\mathcal{A}$ braided with $F: \mathcal{A}^{bop} \xrightarrow{\otimes, br} Z(\mathcal{C})$ faithful

$Z_{\mathcal{A}}(\mathcal{C}) :=$ Full subcat of $Z(\mathcal{C})$ That centralizes The image of $\mathcal{A} \to Z(\mathcal{C})$

[Laugwitz-Walton] $\mathcal{A}$ non-deg, $F(\mathcal{A}^{bop})$ closed by $\oplus$ and subquot.

$$\text{If } \mathrm{SqrT}_{\mathcal{C}}(D, \zeta_D) \neq \emptyset \Rightarrow Z_{\mathcal{A}}(\mathcal{C}) \text{ is modular}$$

Application: Modularity of braided Drinfeld doubles:

Take $\mathcal{A} = K\text{-mod}$ for K quasi Δ Hopf

$\mathcal{C} = H\text{-mod}(\mathcal{A})$ for H Hopf. alg in $\mathcal{A}$.

Then $Z_{\mathcal{A}}(\mathcal{C}) \simeq D_K(H, H^*)\text{-mod}$

"Braided Drinfeld double"

Quantum groups are of This form.! Next: get more examples...

Nichols algebras

For a matrix $\mathbf{q}=(q_{ij})$ in $\Bbbk^\times$ $\rightsquigarrow$ Nichols algebra $\mathcal{B}_{\mathbf{q}}$, braided Hopf algebra

[Heckenberger] Classification of $\mathbf{q}$ s.t. $\dim \mathcal{B}_{\mathbf{q}} < \infty$ } Using gen.^{ed} Root systems and Weyl groupoids!

[Andruskiewitsch-Angiono] Lie-Theoretical organization

- Cartan $\rightsquigarrow$ Frobenius-Lusztig kernels
- Super $\rightsquigarrow$ contragredient Lie superalg / $\mathbb{C}$.
- Supermodular $\rightsquigarrow$ " " " in char >0
- UFO $\rightsquigarrow$ Unidentified.

Realization of $q=(q_{ij})$: a (finite, abelian) group G with $\begin{cases} \text{Elt's } g_i \in G \\ \text{Chars } \chi_j \in \widehat{G} \end{cases}$ s.t. $\chi_j(g_i) = q_{ij}$

Usually: Consider $B_q \in \mathrm{Hopf}({}^{G}_{G}\mathcal{YD})$ $\rightarrow$ Yetter-Drinfeld mods $\rightsquigarrow$ $\mathrm{Drin}(B_q \rtimes G^*)$

Instead: Consider $B_q, B_q^* \in \mathrm{Hopf}(\mathcal{A}_q)$ $\rightarrow$ $\mathcal{A}_q = G\text{-comod}$ with (dual) R-matrix

$$r(g_i \otimes g_j) = q_{ji}$$

Can define $\mathrm{Drin}_{G^*}(B_q^*, B_q)$!

Double vs. Rel double: $\exists$ $\mathrm{Drin}(B_q \rtimes G^*) \overset{\varphi}{\twoheadrightarrow} \mathrm{Drin}_{G^*}(B_q^*, B_q)$

$$\begin{tikzcd}
{}^{B_q}_{B_q}\mathcal{YD}(\mathcal{A}_q) \arrow[r, hook] \arrow[d, "\wr"] & {}^{B_q \rtimes G^*}_{B_q \rtimes G^*}\mathcal{YD} \arrow[d, "\wr"] \\
\mathrm{Drin}_{G^*}(B_q^*, B_q)\text{-mod} \arrow[r, "\mathrm{Res}_\varphi"] & \mathrm{Drin}(B_q \rtimes G^*)\text{-mod}
\end{tikzcd}$$

Why Care?

Example: $(d_i a_{ij})$ symmetrized finite Cartan matrix, q root of unity

For $q_{ij} := q^{d_i a_{ij}}$ we get $\mathrm{Drin}_{G^*}(B_q^*, B_q) \simeq u_q(\mathfrak{g})$ small quantum group

Cartan Type

⇝ Majid, Sommerhäuser, Laugwitz

[Laugwitz-Walton] ⇝ Give conditions for modularity of $\mathrm{Drin}_{G^*}(B_q^*, B_q)$
In terms of top-degree of B_q and
non-degeneracy of $\mathcal{H}_q$

Recover known results for modularity of quantum groups

Verify modularity for one example of type super

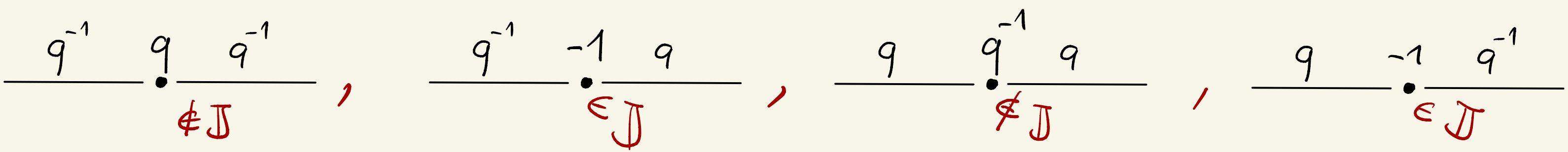
[Laugwitz-S]: Study modularity and applications for q of type super A

Nichols algebras of Type super A:

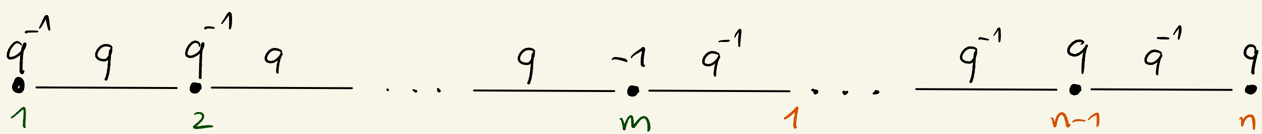
Denoted $A_r(q, \mathbb{J})$

- r: Rank
- q: Root of unity order $N > 2$
- $\mathbb{J}$: Subset of $\{1, \dots, r\}$ "odd roots" (non-empty)

Dynkin diagram: linear concatenation of



Example: (q even order)



Here $\mathcal{B}_{\mathbf{q}} \simeq U_q^+(\mathfrak{sl}(m|n))$

Quantum groups of Type Super A: Assume N = order of q is even

We define a small quantum group $U_q(\mathfrak{sl}(r,\mathbb{J}))$ by gens and rels, and:

- Show that $U_q(\mathfrak{sl}(r,\mathbb{J})) \simeq \mathrm{Drin}_{G^*}(B_q^*, B_q)$
- Provide Triangular decomposition $U_q(\mathfrak{sl}(r,\mathbb{J})) = B_q^* \otimes \Bbbk G \otimes B_q$ and PBW-basis
- Classify simples in $\mathcal{C}$ (h. weights, after Bellamy-Thiel)

Theorem [LS] For $\mathcal{C} = U_q(\mathfrak{sl}(r,\mathbb{J}))$-mod of Type $A_r(q,\mathbb{J})$, TFAE:

1) $\mathcal{C}$ is ribbon.
2) $\mathcal{C}$ is modular.
3) All roots are odd, i.e. $\mathbb{J} = \{1, \dots, r\}$

- For rank $r=2$, $\rightsquigarrow$ explicit description of simples and their dimensions
  $\rightsquigarrow$ invariants of knots
- For rank $r=2$ at 4^{th} root of unity $\rightsquigarrow$ Tensor products of simples

Definition: $U_q(\mathfrak{sl}(r,\mathbb{J}))$ is defined by gens x_i, y_i, κ_i $(1 \le i \le r)$ and rels

$$\kappa_i x_i = q x_i \kappa_i, \qquad \kappa_i x_j = x_j \kappa_i, \quad (i \neq j) \qquad \kappa_i y_i = q^{-1} y_i \kappa_i, \qquad \kappa_i y_j = y_j \kappa_i, \quad (i \neq j)$$

$$\kappa_i \kappa_j = \kappa_j \kappa_i, \qquad \kappa_i^N = 1 \qquad ,$$

$$y_i x_j - q^{u_{ji}} x_j y_i = \delta_{ij}(1 - \overline{\gamma}_i \gamma_i)$$

$$x_{ij} = 0 \quad (i < j-1), \qquad x_{iii\pm1} = 0 \quad (i \notin \mathbb{J}), \qquad x_i^2 = 0 \quad (i \in \mathbb{J}),$$

$$[x_{(i-1i+1)}, x_i] = 0 \quad (i \in \mathbb{J}), \qquad x_{(ij)}^N = 0 \quad (\alpha_{ij} \text{ even root}),$$

$$y_{ij} = 0 \quad (i < j-1), \qquad y_{iii\pm1} = 0 \quad (i \notin \mathbb{J}), \qquad y_i^2 = 0 \quad (i \in \mathbb{J}),$$

$$[y_{(i-1i+1)}, y_i] = 0 \quad (i \in \mathbb{J}), \qquad y_{(ij)}^N = 0 \quad (\alpha_{ij} \text{ even root}), .$$

Example: Rank $r=2$, $\mathbb{J} = \{1,2\}$, $N = 2n$

$$x_1^2 = x_2^2 = 0, \qquad (x_1 x_2)^N = -(x_2 x_1)^N, \qquad y_1^2 = y_2^2 = 0 \qquad (y_1 y_2)^N = -(y_2 y_1)^N,$$

$$y_1 x_1 + x_1 y_1 = 1 - \kappa_2, \quad y_2 x_2 + x_2 y_2 = 1 - \kappa_1, \quad y_2 x_1 = q x_1 y_2, \quad y_1 x_2 = x_2 y_1.$$

The coproduct is determined by

$$\Delta(x_1) = x_1 \otimes 1 + \kappa_1^n \kappa_2 \otimes x_1, \qquad \Delta(x_2) = x_2 \otimes 1 + \kappa_2^n \otimes x_2,$$

$$\Delta(y_1) = y_1 \otimes 1 + \kappa_1^n \otimes y_1, \qquad \Delta(y_2) = y_2 \otimes 1 + \kappa_1 \kappa_2^n \otimes y_2.$$

Rep Theory

Have a Triangular decomp. $\overbrace{U_q(\mathfrak{sl}(r,\mathbb{J}))}^{U} \simeq \overbrace{B_q^* \otimes}^{U^-} \underbrace{K \otimes B_q}_{U^+}$

Can follow highest weight Theory yoga

Lattice $\Lambda = \left(\mathbb{Z}/_{N}\mathbb{Z}\right)^{\times r}$, $\lambda \in \Lambda \rightsquigarrow \Bbbk_\lambda \in K\text{-mod}$

Standard $M(\lambda) := \mathrm{Ind}_{U^+}^{U} \mathrm{Inf}_{K}^{U^+} \Bbbk_\lambda$

Simple head (unique) $M(\lambda) \twoheadrightarrow L(\lambda)$

[Bellamy-Thiel, Vay]: 1) $\{L(\lambda),\ \lambda \in \Lambda\}$ classify isoclasses of simples

2) Projective cover of $L(\lambda)$ has standard filtration

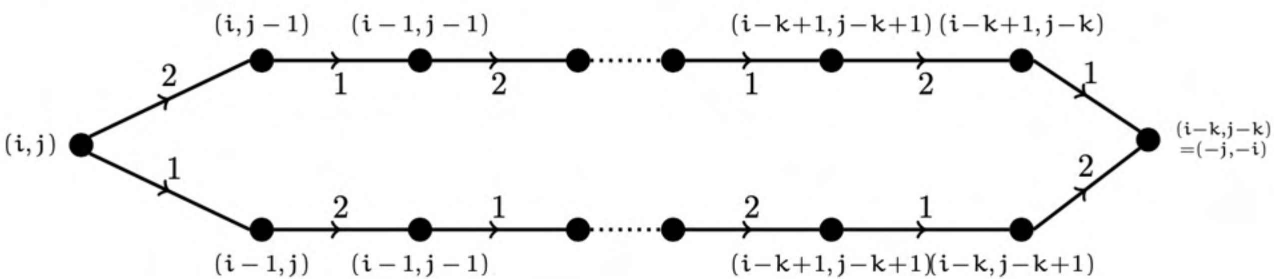
3) Brauer reciprocity ...

Rank 2 case: $\mathbb{J} = \mathbb{I} = \{1,2\}$, q of order $N = 2n > 2$

$$\Lambda = \mathbb{Z}/_N \times \mathbb{Z}/_N$$

Theorem 2.2. *The following is a complete list of non-isomorphic simple* $u_q(\mathfrak{sl}_{2,\mathbb{I}})$*-modules.*

1. $L(0,0) = \mathbf{1}$ *is the tensor unit, the unique simple 1-dimensional module.*
2. *For* $0 < j < N$, $L(0,j)$ *given by*
$$(0,j) \xrightarrow{1} (-1,j) \xrightarrow{2} (-1,j-1) \cdots\cdots (-j+1,1) \xrightarrow{1} (-j,1) \xrightarrow{2} (-j,0)$$
3. *For* $0 < i < N$, $L(i,0)$ *given by*
$$(i,0) \xrightarrow{2} (i,-1) \xrightarrow{1} (i-1,-1) \cdots\cdots (1,-i+1) \xrightarrow{2} (1,-i) \xrightarrow{1} (0,-i)$$
4. *For* $0 < i,j < N$, $L(i,j)$ *given by*



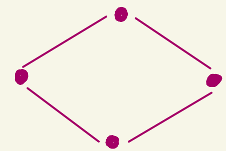
Here, we choose the unique representative $1 \leqslant k \leqslant N$ *of* $i + j$ *modulo* N.

	dim	qdim
1.	1	1
2.	$2j+1$	$(-1)^j$
3.	$2i+1$	$(-1)^i$
4.	$4k$	0

How To obtain knots-invariants

- Using 3-dim module  can use usual Traces

  but get "Trivial" invariant Counts parity of # connected components

- Using 4-dimensional module 

  $N = 2n$

  $L(n, n+1) = W$

Since qdim = 0, need To use modified Traces [Geer–Patureau-Mirand–Turaev]

But get a nice invariant of framed links

RT functor $F_W : \begin{Bmatrix} \text{oriented} \\ \text{Tangles} \end{Bmatrix} \longrightarrow U\text{-mod}$ $(+) \longmapsto W$, $(-) \longmapsto W^*$

$$I_W(\mathcal{L}) = d_W(F_W(\Gamma_{\mathcal{L}}))$$

Ribbon diagram, $F_W(\Gamma_{\mathcal{L}}) \in \mathrm{End}_U(W)$

Ambidextrous Trace $d_W : \mathrm{End}_U(W) \longrightarrow \Bbbk$

Example: $T_{a,b}$ = Torus knot on a-strands braided b-times

(Wind b-times the hole of the Torus, loop a-times around z-axis, join ends)

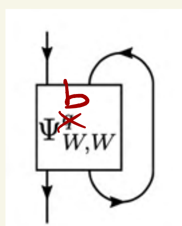
→ Wolfram

$T_{2,3}$ $\quad$ $T_{2,5}$ $\quad$ $T_{2,7}$

For any $b \geq 1$ we have

$$\mathrm{I}_W(\mathcal{T}_{2,b}) = (-1)^{b+1}\left(b + 2\sum_{i=1}^{b-1}(b-i)q^i\right) = (-1)^{b+1}\left(\frac{2}{1-q^{-1}}[b]_q + \frac{b}{1-q}[2]_q\right).$$

Tangle used:



$\Psi_{W,W}$ is the braiding of $W \in U$-mod

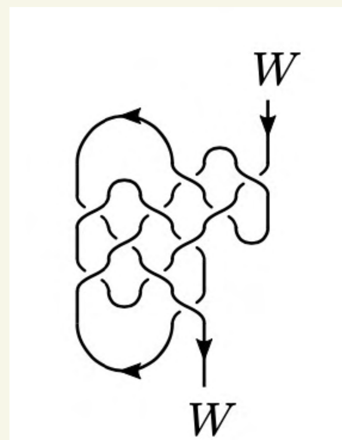
Closed formula for $\mathrm{I}_W(T_{a,b})$?

Example: The knots $T_{2,5} = 5_1$ and 10_{132} have the same Jones, Alexander and HOMFLYPT (different G_2)

$$I_W(5_1) = 2q^4 + 4q^3 + 6q^2 + 8q + 5$$

$$I_W(10_{132}) = 4q^2 + 4q - 3 + 10q^{-2} + 8q^{-3} + 2q^{-4}$$

Tangle used to compute $I_W(10_{132})$



Example [Eliahou-Kauffman-Thistlethwaite]

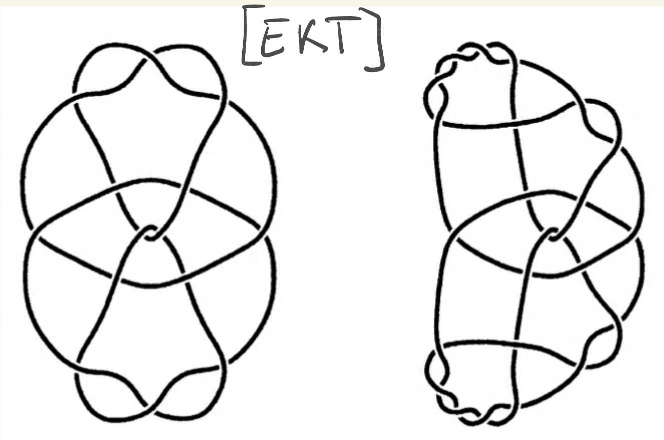
Exist Links $LL_2(k)$, $k \geq 1$, with $\text{Jones}(LL_2(2k)) = \text{Jones}(\text{2-component unlink})$

$$= -t^{-1/2} - t^{1/2}$$

But

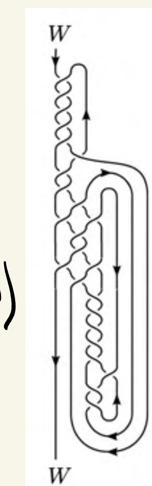
$$\begin{aligned} I_W(LL_2(1)) =& 2\left(q^7 + 2q^6 - q^5 - 3q^4 - 2q^3 + 2q + 2 - q^{-2}\right), \\ I_W(LL_2(2)) =& 2\left(8q^8 + 30q^7 + 49q^6 + 62q^5 + 84q^4 + 77q^3 - 18q^2 - 105q - 99 - 150q^{-1}\right. \\ & \left. -213q^{-2} - 113q^{-3} + 40q^{-4} + 129q^{-5} + 134q^{-6} + 70q^{-7} + 15q^{-8}\right). \end{aligned}$$

[EKT]



$LL_2(1)$ $LL_2(2)$

Tangle used for $I_W(LL_2(2))$



Tensor product of simples

[Bellamy-Thiel, Vay] Exists $K_0(u\text{-mod}^{\mathbb{Z}}) \hookrightarrow \mathbb{Z}\Lambda[\tau,\tau^{-1}]$, $V \longmapsto \mathrm{Ch}^\bullet(V)$

$\mathbb{Z}$-graded modules

Can use it to compute $L(\lambda)\otimes L(\mu)$ in $K_0(u\text{-mod})$

Example: rank $r=2$, $\mathbb{J}=\{1,2\}$, $N=\operatorname{ord} q=4$, $\Lambda=\mathbb{Z}/4\times\mathbb{Z}/4$

$$l_{10}^2 = l_{20} + l_{23}, \qquad l_{10}l_{20} = l_{30} + l_{33}$$

$$l_{10}l_{30} = l_{00} + 2l_{03} + l_{02} + l_{00}, \qquad l_{02}l_{03} = l_{01} + l_{31} + 2l_{20} + l_{10} + l_{01}$$

Gracias !