Problem 1. 1. Let X be a set endowed with the cofinite topology. Prove that a sequence (x_n) converges to a point $x \in X$ if and only if for every $y \neq x$, the set $\{n \in \mathbb{N} \mid x_n = y\}$ is finite.

2. Consider $\mathbb{R}$ with the cofinite topology. Towards what points does the sequence $x_n = \frac{1}{n}$ converge?

Problem 2. Assume that X is a topological spaces with subbasis $\mathcal{S}$, and let $x \in X$. Prove that a sequence (x_n) converges to x if and only if for every $S \in \mathcal{S}$ containing x, there exists some $N \in \mathbb{N}$ such that $x_n \in S$ for every $n \geqslant N$.

Problem 3. Let $\{X_i \mid i \in \Lambda\}$ be a collection of topological spaces, and let (x_n) be a sequence of elements in the Cartesian product $\prod_{i\in\Lambda} X_i$. For each $j \in \Lambda$, let $p_j \colon \prod_{i\in\Lambda} X_i \to X_j$ denote the projection to the j-th coordinate.

1. Endow $\prod_{i\in\Lambda} X_i$ with the product topology. Prove that (x_n) converges to a point $x \in \prod_{i\in\Lambda} X_i$ if and only if for each $j \in \Lambda$, the sequence of j-th coordinates $p_j(x_n)$ in converges to $p_j(x)$ in X_j.

2. Is the statement in Part 2 true if we consider the box topology on $\prod_{i\in\Lambda} X_i$?

Problem 4. In this exercise we endow $\mathbb{R}$ and $\mathbb{R}^2$ with the standard topologies, and quotients with the quotient topology. For the following quotient spaces, describe a homeomorphism to a simpler, known space.

1. $\mathbb{R}/\sim$ where $x \sim y$ if and only if $\text{sgn}(x) = \text{sgn}(y)$.[1]

2. $\mathbb{R}^2/\sim$ where $(x, y) \sim (x', y')$ if and only if $x = x'$.

Problem 5. Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$ be the unit sphere with the induced topology, where $||x|| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. Consider the quotient space $\mathbb{R}P^n = S^n/\sim$ where where $x \sim y$ if and only if $y = x$ or $y = -x$. This space is called real projective space.

1. Find a continuous bijection $\mathbb{R}P^1 \to S^1$.

2. Consider the disk $D^2 = \{x \in \mathbb{R}^2 \mid ||x|| \leqslant 1\}$ with the induced topology from $\mathbb{R}^2$ (note that D^2 contains S^1 as the *boundary* points). Consider on D^2 the equivalence relation $\sim'$ such that $x \sim' -x$ for every $x \in S^1 \subseteq D^2$ (i.e. opposite boundary points of D^2 are identified).

   Find a continuous bijection $\mathbb{R}P^2 \to D^2/\sim'$.

[1] The sign function $\text{sgn} \colon \mathbb{R} \to \mathbb{R}$ is defined by $\text{sgn}(0) = 0$ and $\text{sgn}(x) = \frac{x}{|x|}$ for $x \neq 0$.