On pointed Hopf algebras over non abelian groups with decomposable braiding

Nichols algebras

Let H with bijective antipode and $V \in {}^H_H\mathcal{YD}$.

There exists unique $R = \oplus_{n\geq 0} R^n$ graded Hopf in ${}^H_H\mathcal{YD}$ s.t. $R^0 = \Bbbk$, $\mathcal{P}(R) = R^1 \simeq V$ and $R = \Bbbk\langle R^1\rangle$.

Definition: $R = \mathcal{B}(V)$ is the Nichols algebra of V.

Why care?

Classification of (pointed) Hopf algebras!

Let A s.t. the coradical A_0 is Hopf subalg. Then

- the coradical filtration (A_n) is a Hopf filtration;
- $\operatorname{gr} A = \oplus_{n\geq 0} A_n/A_{n-1}$ is a Hopf algebra;
- $\operatorname{gr} A \simeq R\# A_0$, where $R = (\operatorname{gr} A)^{\operatorname{co} A_0}$ is graded Hopf algebra in ${}^{A_0}_{A_0}\mathcal{YD}$.

Def: $R^1 \in {}^{A_0}_{A_0}\mathcal{YD}$ is the infinitesimal braiding of A.

Then what? Lifting Method!

Assume H fin. dim. cosemisimple.

Goal: classify fin. dim. Hopf algebras A such that the coradical $A_0 \simeq H$ is Hopf subalg. of A.

Suggestion [AS]:

1. Classify $V \in {}^H_H\mathcal{YD}$ such that $\dim \mathcal{B}(V) < \infty$.
2. Obtain a presentation of $\mathcal{B}(V)$.
3. Check if $(\operatorname{gr} A)^{\operatorname{co} A_0} \simeq \mathcal{B}(V)$.
4. Classify A's such that $\operatorname{gr} A \simeq \mathcal{B}(V)\# H$. **"Liftings"**.
   Hint: are all liftings cocycle deformations?

Non abelian groups context

Assume $H = \Bbbk\Gamma$ non abelian group algebra.

[HV]: Classify pairs (V_1, V_2) of abs simple in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ such that $c_{V_2V_1}c_{V_1V_2} \neq \mathrm{id}$ and $\dim \mathcal{B}(V_1 \oplus V_2) < \infty$.

Our problem [HV, Example 1.10]

Let $q_1, q_2 \in \Bbbk^\times$ s.t. $\omega := -q_1q_2$ is 3-th root of 1.

Set $\mathsf{HV}_1 = V_1 \oplus V_2$ the braided vector space with $V_1 = \Bbbk\{x_1, x_2, x_3\}$, $V_2 = \Bbbk\{x_4\}$ and

$$c(x_i \otimes x_j) = -x_{2i-j} \otimes x_i, \quad c(x_4 \otimes x_4) = -\omega^2 x_4 \otimes x_4,$$
$$c(x_i \otimes x_4) = q_1\, x_4 \otimes x_i, \qquad c(x_4 \otimes x_i) = q_2\, x_i \otimes x_4.$$

Our Γ have elements $(g_i)_{i\in\mathbb{I}_4}$ and "1-cocycles" $(\chi_i)_{i\in\mathbb{I}_4}$ satisfying compatibilities $\rightsquigarrow \mathsf{HV}_1 \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$.

Goal: develop lifting method for $\mathcal{B}(\mathsf{HV}_1)$.

2. Presentation of $\mathcal{B}(\mathsf{HV}_1)$

Adjoint action of HV_1 on $T(\mathsf{HV}_1)$:

$$(\operatorname{ad}_c x_i) y = x_i y - (g_i \cdot y) x_i, \quad i \in \mathbb{I}_4,\ y \in T(\mathsf{HV}_1).$$

Notation: $x_{i_1\dots i_n} := (\operatorname{ad}_c x_{i_1}) \dots (\operatorname{ad}_c x_{i_{n-1}}) x_{i_n}$.

Theorem[AnSa]: $\mathcal{B}(\mathsf{HV}_1)$ is minimally presented by generators $(x_i)_{i\in\mathbb{I}_4}$ and relations

$$
\begin{aligned}
x_i^2 &= 0,\ i \in \mathbb{I}_3,\\
x_1x_2 + x_3x_1 + x_2x_3 &= 0,\ x_2x_1 + x_1x_3 + x_3x_2 = 0,\\
x_4^6 &= 0,\\
(x_{124}x_{134} + \omega^2 x_{134}x_{124})^3 &= 0,\\
x_{i14} - \omega x_{1\,2-i\,4} &= 0,\ i \in \mathbb{I}_{2,3},\\
x_4x_{h4} - q_2 x_{h4}x_4 &= 0,\ h \in \mathbb{I}_3.
\end{aligned}
$$

3. Generation in degree one

Theorem[AnSa]: Any pointed f.d. Hopf algebra over Γ with infinitesimal braiding HV_1 is generated by its group-like and skew-primitive elements.

4. Liftings of HV_1 over Γ

We defined a set $\mathcal{R}_{\mathsf{HV}_1} \subset \Bbbk^4$ consisting of 4-uples $\lambda = (\lambda_i)_{i\in\mathbb{I}_4}$ satisfying constraints. For example,

$\lambda_1 = 0$ if $\chi_i^2 \neq \varepsilon$ or $g_i^2 = 1$ for some $i \in \mathbb{I}_3$,

$\lambda_2 = 0$ if $\chi_i\chi_j \neq \varepsilon$ or $g_ig_j = 1$ for some $i \neq j \in \mathbb{I}_3$.

Define $\mathcal{L}(\lambda)$ for any $\lambda \in \mathcal{R}_{\mathsf{HV}_1}$ as

$$
T(\mathsf{HV}_1)\#\Bbbk\Gamma \Big/ \left\langle
\begin{array}{l}
x_4x_{14} - q_2x_{14}x_4\\
x_1^2 - \lambda_1(1 - g_1^2),\\
x_1x_2 + x_3x_1 + x_2x_3 - \lambda_2(1 - g_1g_2),\\
x_{214} - \omega x_{134} - \lambda_2 x_4,\\
x_4^6 - \lambda_3(1 - g_4^6),\\
\mathtt{a}_{124134} - \lambda_4(1 - g_1^{12}g_4^6)
\end{array}
\right\rangle
$$

Theorem[AnSa]: Let $\lambda \in \mathcal{R}_{\mathsf{HV}_1}$. Then

1. $\mathcal{L}(\lambda)$ is a lifting of $\mathcal{B}(\mathsf{HV}_1)$ over $\Bbbk\Gamma$.
2. $\mathcal{L}(\lambda)$ is a cocycle deformation of $\mathcal{B}(\mathsf{HV}_1)\#\Bbbk\Gamma$.

Conversely, if L is lifting of $\mathcal{B}(\mathsf{HV}_1)$ over Γ, there exist $\lambda \in \mathcal{R}_{\mathsf{HV}_1}$ such that $L \simeq \mathcal{L}(\lambda)$.

References

- [AS]Andruskiewitsch, N., Schneider, H.-J. *Lifting of quantum linear spaces and pointed Hopf algebras of order p^3.* J. Algebra 209 (2): 658–691, (1998).
- [AnSa] Angiono, I., Sanmarco, G. *Pointed Hopf algebras over non abelian groups with decomposable braidings, I.* arXiv:1905.04285.
- [HV] Heckenberger, I., Vendramin, L. *The classification of Nichols algebras with finite root system of rank two.* J. Europ. Math. 7, 1977–2017 (2017).

Guillermo Sanmarco, joint work with Iván Angiono

gsanmarco@famaf.unc.edu.ar

Universidad Nacional de Córdoba - CONICET

UNC