Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (1pt)**.** Let G be a bipartite graph with parts U, W, and fix an integer $k \geqslant 1$. Show that if $\deg u \geqslant k$ for all $u \in U$ and $\deg w \leqslant k$ for all $w \in W$, then G has a matching which saturates U.

Problem 2 (2pts)**.** Let G be a graph on $n \geqslant 2$ vertices with $\delta(G) \geqslant n/2$. Show that:

(a) If n is even, then G has a perfect matching.

(b) If n is odd, then $G - v$ has a perfect matching for every $v \in V(G)$.

Hint: Hamiltonian cycles.

Problem 3 (2pts)**.** Let G be a k-regular bipartite graph. Prove that we can partition the edges of G into k perfect matchings. That is, show that we can can partition $E(G) = M_1 \sqcup \cdots \sqcup M_k$ where each M_i is a perfect matching in G.

Problem 4 (2pts)**.** Let G be a bipartite graph with parts U, W wherein no vertex of U is isolated. Show that if all vertices in U have distinct degrees, then G contains a matching which saturates U.

Problem 5 (1+2pts)**.** Let G be an n-vertex graph. Recall that the **degeneracy** of a graph G is

$$d(G) = \max\{\delta(H) \colon H \text{ is a subgraph of } G\}$$

(a) Show that there is an ordering $V(G) = \{v_1, \ldots, v_n\}$ such that $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leqslant d(G)$ for all $i \in \{1, \ldots, n\}$.

Hint: induct by deleting an appropriate vertex.

(b) Use the ordering on part (a) to give an alternative proof of Theorem 10.9, i.e. $\chi(G) \leqslant 1 + d(G)$.

Problem 6 (1 bonus pt)**.** Let G be an n-vertex graph. Show that $d(G)$ is the smallest integer d such that there is an ordering $V(G) = \{v_1, \ldots, v_n\}$ so that $|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \leqslant d$ for all $i \in \{1, \ldots, n\}$.

This problem will be graded all-or-nothing.