Last time:

- For positive integers m, n, the **Ramsey number** $R(m, n)$ is the smallest integer N such that every red,blue-coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n.
- To prove that $R(m, n) \geqslant N$, we must **construct a** red,blue-coloring of $E(K_{N-1})$ which has neither a red copy of K_m nor a blue copy of K_n.
- To prove that $R(m, n) \leqslant N$, we must **show that every** red,blue- coloring of $E(K_N)$ contains either a red copy of K_m or a blue copy of K_n.
- $R(n, m)$ exists for any n, m.
- $R(3, 3) = 6$.
- $R(4, 3) = 9$.
- $R(n, n) \leqslant 4^n$.

Theorem. $R(4,4) = 18$.

Best available bounds:

$$43 \leqslant R(5,5) \leqslant 48, \qquad 102 \leqslant R(6,6) \leqslant 165.$$

Theorem. $R(n,n) \geqslant \sqrt{2}^n$ *for any* $n \geqslant 2$*,*

Definition. *Given* k *positive integers* $n_1, \ldots, n_k$*, the* ***Ramsey number*** $R_k(n_1, \ldots, n_k)$ *is the minimum* N *such that any coloring of* $E(K_N)$ *using* k *colors contains a clique of size* n_i *in color* i*, for some* i*.*

Theorem. *For any positive integers* $n_1, \ldots, n_k$*, we have*

$$R_k(n_1, \ldots, n_{k-2}, n_{k-1}, n_k) \leqslant R_{k-1}(n_1, \ldots, n_{k-2}, R(n_{k-1}, n_k)).$$

In particular, the Ramsey number $R_k(n_1, \ldots, n_k)$ *exists.*

Theorem. $R(3,3,3) = 17$.

Theorem (Schur, 1916). *For any* $k \geqslant 2$*, there is* $N > 3$ *such that for any* k*-coloring of* $\{1, 2, \ldots, N\}$*, there are three integers* x, y, z *of the same color such that* $x + y = z$*.*

Theorem. *For each* $m \geqslant 1$, *there is* p_0 *such that for any prime* $p \geqslant p_0$, *the congruence*

$$x^m + y^m \equiv z^m \mod p$$

has a solution