Definition. *If* G *is a graph with vertices* $V(G) = \{v_1, \ldots, v_n\}$, *the **adjacency matrix** of* G *is the* $n \times n$ *matrix*

$$A = [a_{ij}], \quad \textit{where} \quad a_{ij} = \begin{cases} 1 & \textit{if } v_i v_j \in E(G), \\ 0 & \textit{otherwise.} \end{cases}$$

- The adjacency matrix is symmetric.
- The diagonal entries are all ________.
- The number of 1's on the i-th column equals ________.

Example. *Compute the adjacency matrix of the following graph*

Theorem. *For each positive integer* k*, and each* $i, j \in \{1, \ldots, n\}$*, the* (i, j)*-th entry of the matrix* A^k *counts the number of* v_i–v_j *walks of length exactly* k*.*

Previously: Two graphs G, H are equal if $V(G) = V(H)$ and $E(G) = E(H)$.

Definition. • *An **isomorphism** from* G *to* H *is a bijective function* $\phi\colon V(G) \to V(H)$ *such that*

$$uv \in E(G) \text{ if and only if } \phi(u)\phi(v) \in E(H).$$

• *If there is an isomorphism* ϕ *as above, we say that* G *and* H *are **isomorphic**, and denote* $G \cong H$ *or* $\phi\colon G \xrightarrow{\cong} H$.

Example. *Show that the following graphs are isomorphic.*

Yoga: If $G \cong H$, then we can translate any property or construction from G to H (and vice versa).

Theorem. *Assume* $\phi\colon G \xrightarrow{\cong} H$ *is an isomorphism. Then*

(a) $|V(G)| = |V(H)|$ *and* $|V(H)|$.

(b) $(v_0, v_1, \ldots, v_n)$ *is a {path, walk, trail, cycle} in* G *iff* $(\phi(v_0), \phi(v_1,) \ldots, \phi(v_n))$ *is a {path, walk, trail, cycle} in* H.

(c) G *is connected iff* H *is connected.*

(d) G *is bipartite iff* H *is bipartite.*

(e) For any $v \in V(G)$, $N_H(\phi(v)) = \{\phi(u)\colon u \in N_G(u)\}$, *i.e.,* ϕ *induces a bijection from* $N_G(v)$ *to* $N_H(\phi(v))$.

(f) For any $v \in V(G)$, $\deg_G(v) = \deg_H(\phi(v))$. *Thus* G *and* H *have the same degree sequences.*

(g) $\delta(G) = \delta(H)$ *and* $\Delta(G) = \Delta(H)$. *In particular,* G *is* r*-regular iff* H *is* r*-regular.*

Example. *Decide (with justification) whether the following graphs are isomorphic.*

Example. *Decide (with justification) whether the following graphs are isomorphic.*

Recall: A binary relation R on a set X is an **equivalence relation** if it is:

(a) **reflexive:** xRx for all x in X;

(b) **symmetric:** if xRy then yRx for all $x, y \in X$;

(c) **transitive:** if xRy and yRz then xRz for all $x, y, z \in X$.

Theorem. *The isomorphism relation* GRH *iff* $G \cong H$ *defines an equivalence relation on the set of all graphs.*

Definition. *An **automorphism** of a graph* G *is an isomorphism from* G *to itselft,* $\mathrm{Aut}(G)$ *denotes the set of automorphisms of* G.

- $\mathrm{Aut}(G)$ is actually a **group** under composition.
- The operation $G \mapsto \mathrm{Aut}(G)$ relates graph theory to group theory.

Example. *Exhibit automorphisms of the following graph.*