21. Monday, August $5^{th}$. Compacteness

**Proposition 20.4** If $X$ is Hausdorff and $Y \subseteq X$ is a compact subspace, Then $Y$ is closed in $X$

**Proof:** Let us show That $X \setminus Y$ is open, i.e, $X \setminus Y$ is a neighborhood of each of its points.

Fix $x_0 \in X \setminus Y$. WTS: $\exists$ $V$ open with $x_0 \in V \subseteq X \setminus Y$.
Now, since $X$ is Hausdorff, for each $y \in Y$ we have $x_0 \neq y$, Therefore There exist open sets $V_y$, $U_y$ with $x_0 \in V_y$, $y \in U_y$ and $V_y \cap U_y = \emptyset$.

Note That $Y \subset \bigcup_{y \in Y} U_y$. Since $Y$ is compact, This cover admits a finite subcover

Thus There are $y_1, \dots, y_n \in Y$ such That $Y \subseteq U_{y_1} \cup \dots \cup U_{y_n}$.

Now we Take $V = V_{y_1} \cap \dots \cap V_{y_n}$. We verify it Satisfies The required properties:

1) $V$ is open, because it is the intersection of finitely many open sets in $X$.

2) $V$ contains $x_0$, because each $V_y$ does for any $y \in Y$.

3) $V \cap Y = \emptyset$, Thus $V \subseteq X \setminus Y$

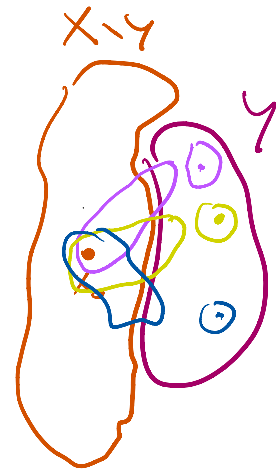
In fact, if $z \in Y$, since $Y \subseteq U_{y_1} \cup \dots \cup U_{y_n}$, we have $z \in U_{y_j}$ for some $j$

Therefore $z \notin V_{y_j}$ because $U_{y_j} \cap V_{y_j} = \emptyset$. Hence $z \notin V$

This proofs that $X \setminus Y$ is a nbhd of every $x_0 \in X \setminus Y$.

Therefore $X \setminus Y$ is open.

Sketch of the proof



**Corollary!** Let $f: X \to Y$ be a bijective continuous map.

If $X$ is compact and $Y$ is Haussdorff, then $f$ is a homeomorphism.

**Proof:** Only need to prove that $f$ is open, which is equivalent to $f$ being closed.
Take $C \subseteq X$ closed. Since $X$ is compact, by prop 20.3, $C$ is compact.
Hence, by Prop 20.2, $f(C) \subseteq Y$ is compact. By prop 20.4, since $Y$ is Hausdorff,
C is closed in $Y$. $\square$

**Examples.** Let $\exp: [0,1] \to S^1$, $\exp(t) = (\cos 2\pi t, \sin(2\pi t))$

1) We proved $\exp: [0,1) \to S^1$ is continuous bijection but not open. Here $S^1$ is Hausdorff, however $[0,1)$ is not compact.

2) We proved $\overline{\exp}: [0,1]/_{\{0,1\}} \longrightarrow S^1$ is a continuous bijection and claimed that hence it is a homeomorphism. Now we know that's true, because $[0,1]/_{\{0,1\}}$ is compact and $S^1$ is Hausdorff.

3) Same argument applies e.g to HW 5.5. Your maps are homeomorphism

So for example $S^1 \approx \mathbb{RP}^1$ and $\mathbb{RP}^2 \approx D^2/\{x \sim -x \text{ for } x \in S^1\}$

**Theorem 2:** A product $X \times Y$ is compact if and only if $X$ and $Y$ are compact

To prove it, we will need the following:

**The Tube Lemma:** Consider $X \times Y$ where $Y$ is compact. Fix $x_0 \in X$

If $N \subseteq X \times Y$ is an open set containing $\{x_0\} \times Y$, then there exists

an open set $U$ of $X$ containing $x_0$ such that $U \times Y \subseteq N$

**Proof:** Since $N \subseteq X \times Y$ is open and contains $\{x_0\} \times Y$, for each

$y \in Y$ there exist open sets $U_y \subseteq X$ and $V_y \subseteq Y$ with $(x_0, y) \in U_y \times V_y \subseteq N$.

Since $Y = \bigcup_{y \in Y} V_y$ and $Y$ is compact, we have $Y = V_{y_1} \cup \dots \cup V_{y_n}$ for some $y_i$'s.

Consider $U = U_{y_1} \cap \dots \cap U_{y_n}$. We prove it satisfies the desired properties

1) $U \subseteq X$ is open, because intersection of finitely many open sets in $X$.

2) $U$ contains $x_0$ because each $U_y$ does.

3) $U \times Y \subseteq N$

Given $(u,y) \in U \times Y$, we have $y \in Y = V_{y_1} \cup \dots \cup V_{y_n}$, so $y \in V_{y_j}$ for some $j$

Also, since $u \in U$, we have $u \in U_{y_j}$. Hence $(u,y) \in U_{y_j} \times V_{y_j} \subseteq N$

Hence $U$ satisfies the desired properties

Sketch: