Definition. *In a connected graph* G*, the* ***distance*** *between vertices* u, v *is*

$$d(u, v) \coloneqq \text{ smallest length of any } u\text{–}v \text{ path in } G.$$

- A u–v path of length $d(u, v)$ is called a **geodesic**.
- The **diameter** of G, denoted $\operatorname{diam}(G)$, is the greatest distance between any two vertices.

Note 1: If G is connected, there are vertices u, v such that $d(u, v) = \operatorname{diam}(G)$.

Example (Exercise 1.12)**.** *For the depicted graph* G*, give an example of each of the following or explain why no such example exists.*

1. *An* x–y *walk of length* 6.
2. *A* v–w *trail that is not a* v–w *path.*
3. *An* r–z *path of length* 2.
4. *An* x–z *path of length* 3.
5. *An* x–t *path of length* $d(x, t)$.
6. *A geodesic whose length is* $\operatorname{diam}(G)$.

Notation:

- If G is a graph and $v \in V(G)$, then $G - v$ is the graph obtained by removing v along with any incident edges.
- If G is a graph and $e \in E(G)$, then $G - e$ is the graph obtained by removing e from G (keep all vertices).
- If G is a graph and $e \notin E(G)$, then $G + e$ is the graph obtained by adding the edge e to G .

Theorem (Theorems 1.8, 1.9)**.** *Let* G *be a graph in at least* 3 *vertices. Then* G *is connected if and only if it contains two distinct vertices* u *and* v *such that* $G - u$ *and* $G - v$ *are connected.*

Definition. *For $n \geqslant 1$, the **path** P_n is the graph with vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$ for $i \in \{1, \ldots, n-1\}$.*

Note that $\text{size}(P_n) = n - 1$.

Definition. *For $n \geqslant 3$, the **cycle** C_n is the graph obtained from P_n by adding the edge $v_n v_1$.*

Note that $\text{size}(C_n) = n$. An {**even, odd**} **cycle** is one of {even, odd} size.

Definition. *For $n \geqslant 1$ the **complete graph** (or **clique**) K_n is the graph on n-vertices, where every two vertices are adjacent.*

Note that $\text{size}(K_n) =$

Next: graphs whose vertices can be partitioned in special ways. As a warm up:

Definition. *An **independent set** on a graph G is a subset X of $V(G)$ such that for any x, y in X, xy is **not** in $E(G)$.*

- **Alternative defn:** the induced subgraph $G[X]$ has ________________
- An independent subset is **maximal** if it is **not** properly contained in another independent subset.

Example ((Maximal) Independent subsets of cycles)**.**

Proposition. *For n odd, the largest size of an independent subset for the cycle C_n is $\frac{n-1}{2}$ (achieved e.g. by $\{v_i : i \text{ is odd}, i \neq n\}$).*

Definition. *A graph* G *is* ***bipartite*** *if there is a partition* $V(G) = X \sqcup Y$ *where* X *and* Y *are independent subsets.*

- If $V(G) = X \sqcup Y$ is a bipartition and $H \subseteq G$ is a subgraph, then $V(H) = (V(H) \cap X) \sqcup (V(H) \cap Y)$ is a bipartition of H.
- A graph is bipartite if and only if each of its components is bipartite.

Example.

Example (Cycles)**.**

Lemma. *No odd cycle is bipartite. All even cycles are bipartite.*

Theorem (Theorem 1.12)**.** *A nontrivial graph* G *is bipartite if and only if* G *contains no odd cycles.*