Name: ____.

Student number: ____.

Instructions

- Turn off all the electronic devices.
- This is a closed book exam.
- You are allowed a notesheet. You can use any fact in the notesheet, but you must reference which fact you are using.
- Unless otherwise stated, you must justify your answers.
- If you have a question, raise your hand and I will come to you.
- You have 50 minutes to complete the exam.
- Good luck!

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

1. (10 points) Let X be a topological space and A, B subsets of X. Determine which statements below are true, and which are false. (If your answer is True, you must prove it; if your answer is False, you must provide a counter example.)

   (a) $(A \cup B)^\circ = A^\circ \cup B^\circ$

   (b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

2. (10 points) Let (X, d_X) and (Y, d_Y) be metric spaces. And consider the set $Z = X \times Y$. Define a function $d_Z \colon Z \times Z \to \mathbb{R}$ by

$$d_Z\big((x, y),\ (x', y')\big) = d_X(x, x') + d_Y(y, y').$$

   (a) Prove that d_Z is a metric on Z.

(b) Let $\mathcal{T}$ be the topology on Z induced by the metric d_Z, and $\mathcal{T}'$ be the product topology on $Z = X \times Y$. Show that $\mathcal{T} = \mathcal{T}'$.

3. (10 points) Let X be a topological space. Let $x \in X$ and let A be a subset of X.

   (a) Prove that $x \in \overline{A}$ if and only if every open set U containing x intersects A

   (b) We say x is a *cluster point* of A if, for every open set U containing x, we have $A \cap (U \backslash \{x\}) \neq \varnothing$.
   The set of all cluster points of A is denoted by A'.
   Prove that $\overline{A} = A \cup A'$.

4. (10 points) Let $\omega \notin \mathbb{R}$ and define $X = \mathbb{R} \cup \{\omega\}$. For each $x \in X$ and $r > 0$, define

$$A(x, r) = \begin{cases} \{y \in \mathbb{R} \mid |y - x| < r\}, & \text{if } x \in \mathbb{R} \\ \{y \in \mathbb{R} \mid |y| > r\} \cup \{\omega\}, & \text{if } x = \omega. \end{cases}$$

(a) Show that $\mathcal{A} = \{A(x, r) \mid x \in X,\ r > 0\}$ is a basis for a topology on X.

(b) Provide X with the topology generated by $\mathcal{A}$ and let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that f extends to a continuous function $g\colon X \to \mathbb{R}$ if and only if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist in $\mathbb{R}$ and are equal.