Last week:

- Vertex coloring
- Chromatic numbers
- Critical graphs
- Chromatic numbers and trees

This week:

- Edge coloring
- Factorization

Definition. *Let* G *be a graph.*

- *An* ***edge-coloring*** *of* G *is a function* $f\colon E(G) \to C$ *where* C *is a non-empty set.*
- *Such edge-coloring is* ***proper*** *if* $f(e) \neq f(s)$ *whenever* $e, s \in E(G)$ *share a common vertex.*
- *For an integer* $t \geqslant 1$*, a* t***-edge-coloring*** *of* G *is a proper coloring* $f\colon E(G) \to C$ *such that* $|C| \leqslant t$*.*
- *The* ***edge-chromatic number*** $\chi'(G)$ *is the smallest integer* t *such that* G *has a proper* t*-edge-coloring.*

Example.

Quick observations:

(a) If G is an n-vertex graph, then $\chi'(G) \leqslant \binom{n}{2}$.

(b) $\chi'(G) = 0$ if and only if G has no edges.

(c) If H is a subgraph of G, then $\chi'(H) \leqslant \chi'(G)$.

Proposition. *Proper edge-colorings of* G *using* t*-colors correspond to partitions of* $E(G)$ *into* t *matchings.*

In particular, $\chi'(G)$ *is the smallest integer for which we can partition* $E(G)$ *into* t *matchings.*

Theorem. *For any graph* G *we have*

$$\chi'(G) \geqslant \Delta(G), \qquad \textit{and} \qquad \chi'(G) \geqslant \frac{E(G)}{\alpha'(G)}.$$

Definition. *For a graph* $G = (V, E)$, *the* ***line graph*** $L(G)$ *has vertex set* E *and edges* $\{es\colon e \neq s \in E, e \cap s \neq \varnothing\}$

In other words, for each edge of G we have a vertex in $L(G)$, and for two edges in G with a common vertex, we have an edge in $L(G)$ between the corresponding vertices.

Example.

Exercise. *(a) Show that* $L(C_n) \simeq C_n$ *for any* $n \geqslant 3$.

(b) Show that $L(P_n) \simeq P_{n-1}$ *for any* $n \geqslant 2$.

(c) For $uv \in E$, *show that* $N_{L(G)}(uv) = \{uw\colon w \in N_G(u)\backslash\{v\}\} \sqcup \{wv\colon w \in N_G(v)\backslash\{u\}\}$.

(d) For $uv \in E$, *show that* $\deg_{L(G)} uv = \deg_G u + \deg_G v - 2$.

Theorem. *If* G *is a graph with at least one edge, then* $\chi'(G) \leqslant 2\Delta(G) - 1$.

Theorem (Vizing's Theorem, 10.12)**.** *If* G *is any graph, then* $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.

Theorem. *For any integer* $n \geqslant 2$*, we have* $\chi'(K_n) = \begin{cases} n-1, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd} \end{cases}$