Last week:

- Eulerian trails and circuits
- Hamiltonian paths and cycles

This week:

- Matchings
- Edge- and vertex-covers
- Edge- and vertex- independence and covering numbers $\alpha, \alpha', \beta, \beta'$.

Recall: An independent set of a graph G is a subset A of vertices such that each edge in G has at most one endpoint in A.

Definition. *Let* $G = (V, E)$ *be a graph.*

- *A **matching** on* G *is a subset* $M \subset E$ *such that each* $v \in V$ *is an endpoint of at most one edge in* M.

  Equivalently: *a matching is a set of edges which are vertex-disjoint.*

- *The **size** of a matching is the number of a edges in it.*

- *Let* $A \subset V$. *We say that a matching* M ***saturates*** A *if every vertex in* A *is incident to some edge in* M.

- *A matching is **perfect** if it saturates* V.

Example.

Example. *Seven seniors a, b, c, d, e, f, g apply for positions A, B, C, D, E, F, G according to the following list:*

- a: B, C
- b: A, B, D, F, G
- c: B, E
- d: B, C, E
- e: A, C, D, F
- f: C, E
- g: D, E, F, G

Is it possible for each student to be hired for a job they have applied?

Upshot:

Definition. *Let* $G = (V, E)$ *be a graph and let* $X \subset V$*. The* ***neighborhood*** $N(X)$ *of* X *is*

$$N(X) = \bigcup_{x \in X} N(x) = \{v \in V \colon vx \in E \text{ for some } x \in X\}$$

Definition. *Let* G *be a bipartite graph with parts* U, W *such that* $|U| \leqslant |W|$*.*
We say that G *satisfies the* ***Hall's condition*** *if*

$$|N(X)| \geqslant |X| \qquad \text{for every non empty } X \subset U.$$

Theorem (8.3 Hall's Theorem)**.** *Let* G *be a bipartite graph with parts* U, W *such that* $|U| \leqslant |W|$*. Then* G *contains a matching which saturates* U *if and only if* G *satisfies Hall's condition.*

Case 1: Every proper subset X of U satisfies $|N(X)| > |X|$.

Case 2: There exists a proper subset X of U such that $|N(X)| = |X|$.

Definition. *A **system of distinct representatives (SDR)** for a collection $S_1, \ldots, S_n$ of finite non-empty sets is a family of distinct elements $s_1, \ldots, s_n$ such that $s_i \in S_i$ for each $i \in \{1, \ldots, n\}$.*

Theorem (8.4, Original formulation of Hall's theorem)**.** *A collection $S_1, \ldots, S_n$ of finite non-empty sets has a SDR if and only if for each $k \in \{1, \ldots, n\}$, the union of any k of these sets contains at least k elements.*

Theorem (8.6)**.** *For any* $r \geqslant 1$*, every* r*-regular bipartite graph has a perfect matching.*

Definition. *Let* $G = (V, E)$ *be a graph*

(1) *The* ***independence number*** $\alpha(G)$ *is the size of a largest independent set in* G.

(2) *The* ***edge-independence number*** $\alpha'(G)$ *is the size of a largest matching in* G.

(3) *A* ***vertex-cover*** *of* G *is a subset* $U \subset V$ *such that every edge of* G *has at least one end-point in* U.
The ***vertex-covering number*** $\beta(G)$ *is the size of a smallest vertex-cover of* G.

(4) *An* ***edge-cover*** *of* G *is a subset* $S \subset E$ *such that every vertex of* G *is incident to at least one edge in* S.
The ***edge-covering number*** $\beta'(G)$ *is the size of a smallest edge-cover of* G.

Example.

Note: Let G be a bipartite graph with parts U, W such that $|U| \leqslant |W|$. Then:

- $\alpha'(G) \leqslant |U|$, and
- $\alpha'(G) = |U|$ if and only if G contains a matching which saturates U.

As a consequence, we can rewrite Hall's Theorem:

Theorem (8.3 Hall's Theorem). *Let G be a bipartite graph with parts U, W such that $|U| \leqslant |W|$. Then $\alpha'(G) = |U|$ if and only if G satisfies Hall's condition.*

Exercise. *Consider integers $n \geqslant 3$ and $1 \leqslant s \leqslant t$. Convince yourself of the following:*

- $\alpha(C_n) = \lfloor n/2 \rfloor$, $\alpha(K_n) = 1$, $\alpha(K_{s,t}) = t$.
- $\alpha'(C_n) = \lfloor n/2 \rfloor$, $\alpha'(K_n) = \lfloor n/2 \rfloor$, $\alpha'(K_{s,t}) = s$.
- $\beta(C_n) = \lceil n/2 \rceil$, $\beta(K_n) = n - 1$, $\beta(K_{s,t}) = s$.
- $\beta'(C_n) = \lceil n/2 \rceil$, $\beta'(K_n) = \lceil n/2 \rceil$, $\beta'(K_{s,t}) = t$.

In particular, verify that for any of the graphs G above, $\alpha(G) + \beta(G) = |V(G)|$ and $\alpha'(G) + \beta'(G) = |V(G)|$.