Unless explicitly requested by a problem, do not include sketches as part of your proof. You are free to use the result from any problem on this (or previous) assignment as a part of your solution to a different problem even if you have not solved the former problem.

Problem 1 (2pts)**.** Let G be any graph. Prove that

$$|E(G)| \geqslant \binom{\chi(G)}{2}.$$

Problem 2 (1pt)**.** Let $G = (V, E)$ be any graph and consider a (not necessarily proper) edge-coloring $f\colon E \to \{\text{red}, \text{blue}\}$). Let G_r and be the graph formed by the red edges and let G_b be the graph formed by the blue edges (formally, $G_r = \big(V, f^{-1}(\text{red})\big)$ and $G_b = \big(V, f^{-1}(\text{blue})\big)$). Prove that

$$\chi(G_r) \cdot \chi(G_b) \geqslant \chi(G).$$

Hint: this generalizes the theorem $\chi(G) \cdot \chi(\overline{G}) \geqslant |V(G)|$ from Lecture 20.

Problem 3 (1pt)**.** For every pair of positive integers m, n, construct a graph G with the following properties:

- $|V(G)| = m \cdot n$, and
- $\chi(G) = m$, and
- $\chi(\overline{G}) = n$.

Note: this shows that the equality can be achieved in $\chi(G) \cdot \chi(\overline{G}) \geqslant |V(G)|$.

Problem 4 (2pts)**.** Prove that G is 3-critical if and only if $G \cong C_{2n+1}$ for some positive integer n.

Problem 5 (0.5 + 1.5 pts)**.** Recall that, for a graph G with $V(G) = \{v_1, \dots, v_n\}$, the **Mycielski graph** $\mu(G)$ is obtained from G by adding:

(1) *shadow vertices* $\{u_1, \dots, u_n\}$ and edges $u_i v_j \in E(\mu(G))$ if $v_i v_j \in E(G)$,

(2) a vertex w and edges $w u_i$ for all shadow vertices u_i.

Show that:

(a) If G is triangle-free, then $\mu(G)$ is triangle free.

(b) $\chi(\mu(G)) = \chi(G) + 1$.

Problem 6 (2pts)**.** Let T be any tree on t vertices, let G be any graph on n vertices and let m be a positive integer. Show that if $n \geqslant (t-1)(m-1)+1$, then either G contains a copy of T or $\overline{G}$ contains a copy of K_m (or both).